

(2) Comparison test: $\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = L$ (finite and $\neq 0$)

if finite then v_n will consider the ~~see~~ the power of v_n and then apply p-series test.

(3) D'Alembert Ratio test: if $\sum u_n$ is a +ve term series such that

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{u_{n+1}} \right) = L$$

if $L > 1$, $\sum u_n$ is convergent

$L < 1$, $\sum u_n$ is divergent

$L = 1$, Test fail

(4) Raabe test $\rightarrow$

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = L$$

$L > 1$, $\sum u_n$ is cgt

$L < 1$, $\sum u_n$ is dgt

$L = 1$, $\sum u_n$ Test fail.

(5) Cauchy root test $\rightarrow$

$$\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = L$$

$L < 1$, $\sum u_n$ is cgt

$L > 1$, $\sum u_n$ is dgt

$L = 1$, $\sum u_n$ ~~Test fail~~

Test fail.

(6) Logarithmic Test:

$$\lim_{n \rightarrow \infty} n \cdot \log \left(\frac{u_n}{u_{n+1}} \right) = L$$

$L > 1$, $\sum u_n$ is cgt.

$L < 1$, $\sum u_n$ is dgt.

(7) Leibnitz test $\rightarrow$ An infinite series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot u_n = (u_1 - u_2 + u_3 - u_4 + \dots \infty)$$

$$\lim_{n \rightarrow \infty} u_n > 0$$

* कौन-कौन से test का use लगाए। this form

① P-series test $\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} p \leq 1, \text{div} \\ p > 1, \text{con} \end{cases}$

② Comparison test $\rightarrow$ हर तथा अंश में n का term होना चाहिए।

③ D'Alembert test $\rightarrow$ जब $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ factorial or x के form में दिया होता है लगाएंगे।

④ Raabe's test $\rightarrow$ जब D'Alembert test fail हो जाए तो Raabe's test लगाए।

⑤ logarithmic test $\rightarrow e$ का form given होता है and when D'Alembert goes to fail $\frac{1}{n}$

⑥ Cauchy root test $\rightarrow$ जब $\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$ or n का power हो किसी term के ऊपर।

⑦ Leibnitz test $\rightarrow$ alternative series में लगाए।

first comparison test
second D'Alembert test
third Raabe test
fourth logarithmic test.

* formula of all test.

① p-series test (power series) $\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$
 $p > 1, \text{con}$
 $p \leq 1, \text{div}$

INFINITE SERIES

Sequence $\div$ Sequence is a function whose domain is $\mathbb{N}$, Notation is $\{a_n\}_{n=1}^{\infty}$.

$$a_n = n^{\text{th}} \text{ term} = f(n)$$

① $f(n) = n^2 = a_n$

$$f(1) = 1 = a_1$$

$$f(2) = 4 = a_2$$

$$f(3) = 9 = a_3$$

$$f(4) = 16 = a_4$$

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

* $f(n) = \frac{1}{n}$, $f(1) = \frac{1}{1} = 1$

$$f(2) = \frac{1}{2} = 0.5$$

$$f(3) = \frac{1}{3} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

$$f(4) = \frac{1}{4} =$$

* Series $\div$ If $\{a_n\}$ represents a sequence then $\sum_{n=1}^{\infty} a_n$ is a series.

$$= \sum_{n=1}^{\infty} n^2 = (1 + 4 + 9 + 16 + \dots)$$

③ $S_n = n^{\text{th}} \text{ partial sum} = (a_1 + a_2 + \dots + a_n)$

④ Behavior of a infinite series

4

9

(b) Divergent

(c) oscillating

14

int

15

Cgt

179

2

7	
---	--

91

$$q_2 = \frac{1}{203}$$

$$93 = \frac{1}{3.4}$$

$$1.9m = \frac{1}{m(m+1)}$$

5. Son

at

92

Q?

20

ist: $\sin = 1 = \sin 90^\circ$
 $n \rightarrow \infty$ (ggf)

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$$* (1^2 + 2^2 + 3^2 + 4^2 + \dots \infty)$$

$$a_1 = 1$$

$$a_2 = 4$$

$$a_3 = 9$$

$$a_4 = 16$$

$$S_n = (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

Formula

$$\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n \rightarrow \infty} S_n = \infty \text{ infinite (Dgt)}$$

EX 10

Test - 1

P-Series test (Power series)

An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is said to be Cgt if $p > 1$ and Dgt if $p \leq 1$

$$\therefore \sum_{n=1}^{\infty} n^2 = \sum_{n=1}^{\infty} \frac{1}{n^{-2}} = p = -2$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = p = -2$$

$p \leq 1$ so it is Dgt

EX 11

$$\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots \infty \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\text{by } \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$p = 3$$

$$p > 1$$

Cgt

Test the Convergence

⑨ $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots \infty$

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots$$

~~$$\frac{1}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$~~

$$a_1 = \frac{1}{1 \cdot 3} = \left(\frac{1}{1} - \frac{1}{3} \right)$$

$$a_2 = \frac{1}{3 \cdot 5} = \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$a_3 = \frac{1}{5 \cdot 7} = \left(\frac{1}{5} - \frac{1}{7} \right)$$

~~$$\frac{1}{n(n+2)} = \left(\frac{1}{n} - \frac{1}{n+2} \right)$$~~

$$= \frac{1}{n+2} =$$

$S_n = 1$ finite

$$S_1, S_2, \dots, S_n$$

$$= 1 + (n-1)2$$

$$= 2n-1$$

$$S_1, S_2, S_3, \dots$$

$$a_n = \frac{1}{(2n-1)(2n+1)}$$

$$\frac{A}{(2n-1)} + \frac{B}{(2n+1)}$$

इससे सबसे ज्यादा
question आती है

Test - 1

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① p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 1, C$$

$$p \leq 1, D$$

Test - 2

② Comparison Test: If $\sum u_n$ and $\sum v_n$ be two the term series such that

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{v_n} \right) = L \quad (\text{finite and non-zero})$$

then $\sum u_n$ and $\sum v_n$ behave alike same.

③ Test the Convergence:

$$\left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots \right)$$

Solⁿ $u_n \rightarrow \frac{1}{n(n+1)}$ Given

$$v_n = \frac{1}{n^2 \left(1 + \frac{1}{n} \right)}$$

$$v_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{\frac{1}{n^2 \left(1 + \frac{1}{n} \right)}}{\frac{1}{n^2}} = \frac{1}{1 + \frac{1}{n}} \times \frac{n^2}{1}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{\left(1 + \frac{1}{\infty} \right)} = \frac{1}{1} = 1$$

where it is finite

$$u_n = \frac{1}{n^2} \quad \boxed{p=2} \quad \text{Convergence}$$

Q. Test the convergence.

$$\left(\frac{2}{1 \cdot 3} + \frac{4}{3 \cdot 5} + \frac{6}{5 \cdot 7} + \dots \infty \right)$$

$u_1 \quad u_2 \quad u_n$

=

~~2, 4, 6, 8, \dots~~

$a = 2$
 $d = 2$

$$a_n = a + (n-1)d$$

$$a_n = 2 + (n-1)2 \Rightarrow 2 + 2n - 2$$

$$u_n = \frac{2n}{(2n-1)(2n+1)} = \frac{2n}{4n^2-1}$$

$$u_n = \frac{2n}{n^2 \left(4 - \frac{1}{n^2} \right)}$$

let $v_n = \frac{1}{n^2}$

$$v_n = \frac{1}{n^2 \left(4 - \frac{1}{n^2} \right)}$$

$$\frac{u_n}{v_n} = \frac{2}{4 - \frac{1}{n^2}}$$

ist $n \rightarrow \infty$ $\frac{u_n}{v_n} = \frac{2}{4-0} = \frac{2}{4} = \frac{1}{2}$

finite and non zero

$v_n = \frac{1}{n^2}$ ~~$p=1$~~ (Divergence)

Test 3* (7) D' Alembert ratio test

If $\sum u_n$ is a (+ve) term series such that

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{u_{n+1}} \right) = l$$

then for $l > 1$, $\sum u_n$ is convergent

$l < 1$, $\sum u_n$ is divergence

$l = 1$, Test fails



Test the convergence :-

$$\left(\frac{1}{1^2} + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \dots \infty \right)$$

$$u_n = \frac{n!}{n^2} \quad \frac{u_n}{u_{n+1}} = \frac{n!}{n^2} \times \frac{(n+1)^2}{(n+1)!}$$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^2}$$

~~$$\frac{n!}{n^2} \times \frac{(n+1)^2}{(n+1)!} = \frac{n! (n+1)}{n^2 (n+1)!}$$~~

~~$$\frac{n!}{n^2} \times \frac{(n+1)^2}{(n+1)!} = \frac{n! (n+1)}{n^2 (n+1)!}$$~~

$$\frac{n!(n+1)^2}{n^2(n+1)!}$$

$$= \frac{\cancel{n!} \cdot \cancel{n} \left(1 + \frac{1}{n}\right)^2}{n^2 \cancel{(n+1)!}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{\left(1 + \frac{1}{n}\right)^2}{(n+1)} = \frac{1}{\infty} = 0$$

$$1 < 1$$

divergence



Comparison test for series

$$u_n = \frac{f(n)}{g(n)} \quad \left(\begin{array}{l} \text{rational} \\ \text{polynomial} \end{array} \right)$$

* ~~Comparison~~ D'Alembert ratio test for series

$$u_n = (\text{factorial}) / 2^n \quad \text{or}$$

multiplication of n numbers

~~Comparison~~



Test the Convergence

$$\sum_{n=1}^{\infty} \frac{2^n}{(n+1)\sqrt{n}} = \sum_{n=1}^{\infty} u_n$$

$$\left(\frac{2}{2 \cdot 1} + \frac{2^2}{3 \cdot 2} + \frac{2^3}{4 \cdot 3} + \frac{2^4}{5 \cdot 4} + \dots \right)$$

this is open form of series

~~final answer of question 30-05/18/20~~

$$u_n = \frac{x^n}{(n+1)\sqrt{n}} \quad u_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^n}{\cancel{\sqrt{n}}(n+1)(\cancel{x})^{n+1}} \cdot \frac{(n+2) \cdot \sqrt{1+\frac{1}{n}}}{\cancel{\sqrt{n}}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(x^n (n+2) \cdot \sqrt{1+\frac{1}{n}})}{n(1+\frac{1}{n}) \cdot x \cdot x^n}$$

$$= \frac{(n+2) \sqrt{1+\frac{1}{n}}}{n(1+\frac{1}{n}) \cdot x}$$

$$= \frac{x(1+\frac{2}{n}) \sqrt{1+\frac{1}{n}}}{x(1+\frac{1}{n}) \cdot x} \Rightarrow \frac{(1+\frac{2}{n}) (\sqrt{1+\frac{1}{n}})}{(1+\frac{1}{n}) (\sqrt{1+\frac{1}{n}}) \cdot x}$$

$$\frac{u_n}{u_{n+1}} \Rightarrow \frac{(1+\frac{2}{n})}{\sqrt{1+\frac{1}{n}} \cdot x} \Rightarrow \frac{1+\frac{2}{\infty}}{\sqrt{1+\frac{1}{\infty}} \cdot x} = \frac{1}{1 \cdot x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ if $\frac{1}{x} > 1$, Cgt
if $\frac{1}{x} < 1$, dgt

$\frac{1}{x} = 1$, test fails for $x=1$

P. 10

Examine the convergence of the series

$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots$$

$$S_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-1}{(n+2)^3-1} \quad \text{Generalise formula}$$

$$\Rightarrow \frac{\sqrt{n+1}-1}{(n+2)^3-1} = u_n$$

$$v_n = \frac{1}{n^3}$$

$$v_n = \frac{1}{n^3} \quad v_n = \frac{1}{n^3} \quad v_n = \frac{1}{n^3}$$

$$v_n = \frac{1}{n^{5/2}} \quad \frac{u_n}{v_n}$$

$$= \left(\frac{\sqrt{n+1}-1}{(n+2)^3-1} \right) \times \frac{n^{5/2}}{1}$$

$$\Rightarrow \frac{n^{5/2} \left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{\left(\left(1+\frac{2}{n}\right)^3 - 1 \right)} \quad \left| \begin{array}{l} \frac{5+1}{2} \\ \frac{6}{2} = 3 \end{array} \right.$$

$$\lim_{n \rightarrow \infty} \frac{n^{5/2} \left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} \right)}{\left(\left(1+\frac{2}{n}\right)^3 - 1 \right)} \Rightarrow \frac{\sqrt{1+0} - \frac{1}{\sqrt{0}}}{(1+0)^3 - 0} = \frac{1-0}{1-0} = \frac{1}{1} = 1$$

finite

$$v_n = \frac{1}{n^{5/2}} \quad \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$p = 5/2$$

$p > 1$ Convergent

Tutorial

$$\textcircled{1} \quad \frac{1}{1 \cdot 2} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \dots \infty$$

$$\textcircled{2} \quad \sum_{n=1}^{\infty} \frac{(n+1)}{n(2n-1)}$$

$$\textcircled{3} \quad \sum_{n=1}^{\infty} (\sqrt{n^2+1} - n)$$

$$\textcircled{4} \quad \left(\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots \right)$$

$$\textcircled{5} \quad \left(\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 5} + \frac{x^5}{5 \cdot 3} + \dots \right)$$

$$\textcircled{6} \quad \left(\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \dots \infty \right)$$

$$\textcircled{1} \quad \frac{1}{1 \cdot 2} + \frac{2}{3 \cdot 4} + \frac{3}{5 \cdot 6} + \dots \infty$$

$$\frac{n}{(n-1)(n)} = \frac{1}{(n-1)2} = \frac{1}{4n-2}$$

$$u_n = \frac{1}{4n-2} \quad \text{Let } v_n = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{1}{4n-2} \times \frac{n}{1} = \frac{n}{4n-2} = \frac{1}{4 - \frac{2}{n}}$$

$$n \rightarrow \infty \quad \frac{u_n}{v_n} = \frac{1}{4 - \frac{2}{\infty}} = \frac{1}{4 - 0} = \frac{1}{4}$$

finite value

$$\text{So } v_n = \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad [p=1] \text{ divergent}$$

$$\sum_{n=1}^{\infty} \frac{(n+1)}{n(n-1)}$$

$$u_n = \frac{(n+1)}{n(n-1)}$$

$$v_n = \frac{n}{n^2}$$

$$v_n = \frac{1}{n}$$

$$n \rightarrow \infty \quad \frac{u_n}{v_n} = \frac{(n+1)}{(2n^2 - n)} \times \frac{n}{1}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{n \cdot n \left(1 + \frac{1}{n}\right)}{n^2 \left(2 - \frac{1}{n}\right)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \left(\frac{1 + \frac{1}{\infty}}{2 - \frac{1}{\infty}} \right) = \frac{1+0}{2} = \frac{1}{2} \quad \text{Finite}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^1} \quad p=1 \quad \text{divergent}$$

$$(3) \sum_{n=1}^{\infty} (\sqrt{n^2+1} - n)$$

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1+1}{2+1} = \frac{1+1}{2}$$

$$u_n = \sqrt{n^2+1} - n$$

$$u_{n+1} = \sqrt{(n+1)^2+1} - (n+1)$$

$$u_n = \sqrt{n^2+1} - n$$

$$v_n = \sqrt{n^2}$$

$$v_n = n$$

$$\frac{u_n}{v_n} = \frac{\sqrt{n^2+1} - n}{\sqrt{n^2}}$$

$$= \frac{\sqrt{n^2} \left(\sqrt{1 + \frac{1}{n^2}} - \frac{n}{\sqrt{n^2}} \right)}{\sqrt{n^2}}$$

$$\Rightarrow \frac{\sqrt{n^2} \left(\sqrt{1 + \frac{1}{n^2}} - \frac{n}{\sqrt{n^2}} \right)}{\sqrt{n^2}}$$

$$\sqrt{1 + \frac{1}{n^2}} - \frac{n}{\sqrt{n^2}} \xrightarrow{n \rightarrow \infty} \frac{u_n}{v_n} \sqrt{1 + \frac{1}{n^2}} - 1$$

$$v_n = \frac{1}{2} \quad \Rightarrow \quad \sqrt{1 + \frac{1}{\infty}} - 1$$

$$\boxed{v_n = \frac{1}{2}} \quad \Rightarrow \quad 0 \text{ finite}$$

$p = 1$ divergent

Q(3)

$$\sum_{n=1}^{\infty} (\sqrt{n^2+1} - n)$$

ans)

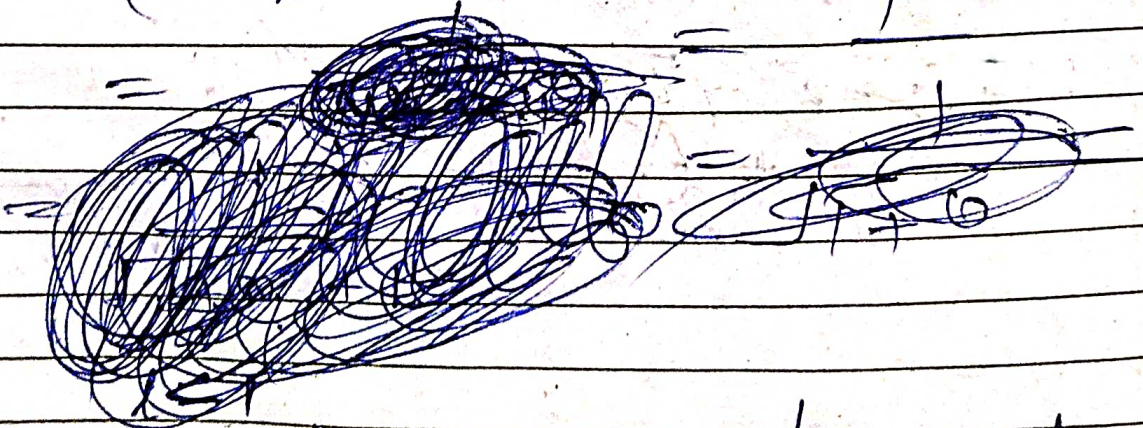
$$u_n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n}$$

$$u_n = \frac{n^2+1 - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n}$$

$$= \frac{1}{n \left(\sqrt{1 + \frac{1}{n^2}} + 1 \right)}$$

$$v_n = \frac{1}{n} \quad \frac{u_n}{v_n} = \frac{1}{\left(\sqrt{1 + \frac{1}{n^2}} + 1 \right)} \quad \times n$$

$$n \rightarrow \infty \quad \frac{u_n}{v_n} = \frac{1}{\left(\sqrt{1 + \frac{1}{n^2}} + 1 \right)} = \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}$$



$$= \frac{1}{\sqrt{1 + \frac{1}{\infty^2}} + 1} = \frac{1}{1 + 1} = \frac{1}{2} \text{ finite}$$

~~test~~ ~~divergent~~ $v = \frac{1}{n}$

$p = 1$ (dgt)

Q5

$$\left(\frac{x}{1 \cdot 3} + \frac{x^2}{3 \cdot 5} + \frac{x^3}{5 \cdot 7} + \dots - \infty \right)$$

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$$u_n = \frac{x}{(2n-1)(2n+1)}$$

$$\frac{u_n}{u_{n+1}}$$

$$\frac{n+1}{x} \cdot \frac{x}{(2n+1)(2n+3)}$$

$$\frac{(2n+1)-1}{(2n+1)+1} = \frac{2n}{2n+2} = \frac{n}{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{x}{(2n-1)(2n+1)} \times \frac{(2n+1)(2n+3)}{x^{n+1}}$$

$$= \frac{x}{(2n-1)(2n+1)} \cdot x = \frac{x^2}{(2n-1)(2n+1)}$$

$$4n^2 + 6n + 2n + 3$$

$$4n^2 - 1 = x$$

$$4n^2 + 8n + 3$$

$$4n^2 - 1 = x$$

$$\frac{u_n}{u_{n+1}} = \frac{x^2 \left(4 + \frac{8}{2n} + \frac{3}{2n^2} \right)}{x^2 \left(4 - \frac{1}{n^2} \right) \cdot x} = \frac{4 + \frac{8}{2n} + \frac{3}{2n^2}}{\left(4 - \frac{1}{n^2} \right) \cdot x}$$

$$4 + 0 + 0$$

$$4 - 0 = x$$

$$\frac{4}{4 \cdot x} = \frac{1}{x}$$

if $\frac{1}{x} > 1$, cgt

if $\frac{1}{x} < 1$, dgt

if $\frac{1}{x} = 1$, Test fail

$$(6) \left(\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \dots \right)$$

$$(6) \frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \dots - \infty$$

$$u_n = \frac{(n+1)!}{3^n}$$

~~$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$~~

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$u_n = \frac{(n+1)!}{3^n}$$

$$u_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!}{3^n} \times \frac{3^{n+1}}{(n+2)!}$$

~~$$= \frac{(n+1)! \times 3}{3 (n+2)(n+1)(n)!}$$~~

$$\frac{3}{n+2} = \frac{3}{n+2} \cdot \frac{3}{3}$$

$$\frac{1}{3} \cdot 3 \quad 0 \times 3 = 0$$

~~$$1 < 1$$~~

④ $\frac{x^0}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots \infty$

$$U_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}, \quad U_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^{2n-2}}{x^{2n}} \times \frac{(n+2)(\sqrt{n+1})}{(n+1)\sqrt{n}}$$

$$\frac{U_n}{U_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n} \cdot x^2} = \frac{\sqrt{n+1}(n+2)}{\sqrt{n}(n+1) \cdot x^2}$$

$$= \frac{\sqrt{n} \left(\sqrt{1 + \frac{1}{n}} \right) \cdot n \left(1 + \frac{2}{n} \right)}{\sqrt{n} \cdot n \left(1 + \frac{1}{n} \right) \cdot x^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{\sqrt{1 + \frac{1}{n}} \cdot \left(1 + \frac{2}{n} \right)}{\left(1 + \frac{1}{n} \right) \cdot x^2} \Rightarrow \frac{\sqrt{1 + \frac{1}{n}} \cdot \left(1 + \frac{2}{n} \right)}{\left(1 + \frac{1}{n} \right) \cdot x^2}$$

$$\frac{U_n}{U_{n+1}} = \frac{(1+0) \cdot (1+0)}{(1+0) \cdot x^2} = \frac{1}{x^2}$$

$$\frac{1}{x^2} > 1 = \text{Cgt}$$

$$\frac{1}{x^2} < 1 = \text{dgt}$$

$$\frac{1}{x^2} = 1 = \text{test fail}$$

*

Raabe's test

If $\sum u_n$ is a +ve term series such that

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \ell$$

Then for $\ell > 1$, $\sum u_n$ is C.T.

$\ell < 1$, $\sum u_n$ is D.T.

$\ell = 1$, $\sum u_n$ Test fails

अगर $\ell = 1$ तो D'Alembert Ratio fails
 $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 1$

(ii) if $\lim_{n \rightarrow \infty} \left(\frac{u_n}{u_{n+1}} \right)$ does not involve ℓ

* # Gauss Test

if $\sum u_n$ is a +ve term series such that

$$\frac{u_n}{u_{n+1}} = \left[1 + \frac{d}{n} + O\left(\frac{1}{n^2}\right) \right]$$

then for $d > 1$, $\sum u_n$ is C.T.

$d \leq 1$, $\sum u_n$ is D.T.

Some Condition of Raabe's test

Q# Test the convergence

$$\sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdot 10 \cdots (3n+1) \cdot x^n}{1 \cdot 2 \cdot 3 \cdots n}$$

$$u_n = \frac{4 \cdot 7 \cdot 10 \cdots (3n+1) x^n}{1 \cdot 2 \cdot 3 \cdots n}$$

$$u_{n+1} = \frac{4 \cdot 7 \cdot 10 \cdots (3n+1)(3n+4) x^{n+1}}{1 \cdot 2 \cdot 3 \cdots n(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{4 \cdot 7 \cdot 10 \cdots (3n+1) x^n \cdot 1 \cdot 2 \cdot 3 \cdots n(n+1)}{1 \cdot 2 \cdot 3 \cdots n \cdot 4 \cdot 7 \cdot 10 \cdots (3n+1)(3n+4) x^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(3n+1)x \cdot n(n+1)}{n \cdot (3n+1)(3n+4)x \cdot x} = \frac{n+1}{(3n+4)x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{n(1 + \frac{1}{n})}{n(3 + \frac{4}{n})x} = \frac{(1 + \frac{1}{n})}{(3 + \frac{4}{n})x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{(3+0)x} = \frac{1}{3x}$$

if $\Rightarrow \frac{1}{3x} > 1$, Cgt

$$\frac{1}{3x} < 1$$

$\frac{1}{3x} = 1$ Test fail

for $x = \frac{1}{3}$, Apply Raabe Test

$$\frac{u_m}{u_{m+1}} = \frac{(n+1)}{(3n+4)x} = \left(\frac{3n+3}{3n+4} \right)$$

$$\lim_{m \rightarrow \infty} n \left(\frac{u_m}{u_{m+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{3n+3}{3n+4} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{3n+3 - 3n-4}{3n+4} \right) = \lim_{n \rightarrow \infty} n \left(\frac{-1}{3n+4} \right) = -\frac{1}{3}$$

$$\lim_{n \rightarrow \infty} n \left(\frac{3n+3}{3n+4} - 1 \right)$$

$$n \left(\frac{3n+3}{3n+4} - 1 \right)$$

$$n \left(\frac{3n+3}{3n+4} - 1 \right) = \frac{n}{n} \left(\frac{3n+3}{3n+4} - 1 \right)$$

$$\frac{3n+3}{3n+4} - 1 = \frac{3n+3 - 3n-4}{3n+4} = \frac{-1}{3n+4}$$

$$- \frac{1}{3} < -1, D$$

Q Test the convergence :-

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{2n!} \cdot x^{2n}$$

$$u_n = \frac{(n!)^2}{2n!} \cdot x^{2n}$$

$$u_{n+1} = \frac{((n+1)!)^2}{2(n+1)!} \cdot x^{2(n+1)}$$

$$u_{n+1} = \frac{((n+1)!)^2}{2(n+2)!} \cdot x^{2n+2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2 \cdot x^{2n}}{2n!} \times \frac{2(n+2)!}{((n+1)!)^2 \cdot x^{2n+2}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n! \cdot n! \cdot (2n+2)(2n+1)(2n)!}{(n+1)(n!)^2 \cdot x^2 \cdot 2n!}$$

$$= \frac{(2n+2)(2n+1)(2n)!}{(n+1)^2 \cdot x^2 \cdot (2n)!}$$

$$= \frac{4n^2 + 6n + 2}{n^2 + 1 + 2n} \cdot x^2$$

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 6n + 2}{(n^2 + 2n + 1)x^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 \left(4 + \frac{2}{n} + \frac{1}{n} + \frac{2}{n^2} \right)}{n^2 \left(1 + \frac{1}{n^2} + \frac{2}{n} \right) \cdot x^2}$$

$$= \frac{\left(4 + \frac{2}{n} + \frac{1}{n} + \frac{2}{n^2} \right)}{\left(1 + \frac{1}{n^2} + \frac{2}{n} \right) x^2}$$

$$= \frac{4}{1 \cdot x^2} = \frac{4}{x^2}$$

$$x^2 < 1, d$$

$$\frac{4}{x^2} > 1, c$$

$$\frac{4}{x^2} = 1 \text{ test failed}$$

for $x^2 = 4$, Apply Raabe's test

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 6n + 2}{(n^2 + n + 1)x^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 6n + 2}{4(n^2 + n + 1)} = \frac{4n^2 + 6n + 2}{4n^2 + 8n + 4}$$

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{4n^2 + 6n + 2 - 4n^2 - 8n - 4}{4n^2 + 8n + 4} \right)$$

$$\lim_{n \rightarrow \infty} n \left(\frac{-2n - 2}{4n^2 + 8n + 4} \right) = \lim_{n \rightarrow \infty} n \left(\frac{-2 - \frac{2}{n}}{4 + \frac{8}{n} + \frac{4}{n^2}} \right)$$

$$\lim_{n \rightarrow \infty} \frac{\left(-2 - \frac{2}{n} \right)}{\left(4 + \frac{8}{n} + \frac{4}{n^2} \right)} = \frac{-2 - 0}{4 + 0 + 0}$$

$$= \frac{-2}{4} = -\frac{1}{2} \because -\frac{1}{2} < 1, \text{ so } d \text{ is true}$$

for $x=1$, Apply Ratio Test

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

$$\text{Now } n \left(\frac{u_n}{u_{n+1}} - 1 \right) = n \left(\frac{3n+7}{3n+3} - 1 \right)$$

$$= n \left(\frac{3n+7-3n-3}{3n+3} \right)$$

$$= n \left(\frac{4}{3n+3} \right)$$

$$\Rightarrow \frac{4n}{3n+3} \Rightarrow \frac{4 \cdot n}{n(3+\frac{3}{n})} \quad \frac{4}{3} \geq 1, C$$

$$= \frac{4}{3} \left(\frac{1}{3+\frac{3}{n}} \right) = \frac{4}{3+\frac{3}{n}} \quad \frac{4}{3} < 1, D$$

$$= \frac{4}{3} = 1.33 > 1$$

$\therefore 1 > 1$ so sum is Cgt

(ii)

$$\frac{1^2}{4^2} + \frac{1 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1 \cdot 5 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \dots \infty$$

Solⁿ:

$$\frac{1^2}{4^2} + \frac{1 \cdot 5^2}{8^2} + \frac{1 \cdot 5 \cdot 9^2}{12^2} + \dots + \frac{(4n+3)^2}{(4n)^2}$$

$$u_n = \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \dots + \frac{(4n+3)^2}{(4n)^2}$$

$$u_m = \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4m-3)}{4^2 \cdot 8^2 \cdot 12^2 \cdot \dots \cdot (4m)^2}$$

$$u_{m+1} = \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4m-3) \cdot (4(m+1)-3)}{4^2 \cdot 8^2 \cdot 12^2 \cdot \dots \cdot (4m)^2 \cdot (4(m+1))^2}$$

$$u_{m+1} = \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4m-3) \cdot (4m+1)}{4^2 \cdot 8^2 \cdot 12^2 \cdot \dots \cdot (4m)^2 \cdot (4m+4)^2}$$

~~$$u_{m+1} = \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4m-3) \cdot (4m+1)}{4^2 \cdot 8^2 \cdot 12^2 \cdot \dots \cdot (4m)^2 \cdot (4m+4)^2}$$~~

~~$$\frac{u_m}{u_{m+1}} = \frac{1 \cdot 5 \cdot 9 \cdot \dots \cdot (4m-3) \cdot 4 \cdot 8 \cdot 12 \cdot \dots \cdot (4m) \cdot (4m+4)}{4^2 \cdot 8^2 \cdot 12^2 \cdot \dots \cdot (4m)^2 \cdot 1 \cdot 5 \cdot 9 \cdot \dots \cdot (4m-3) \cdot (4m+1)^2}$$~~

$$\frac{u_m}{u_{m+1}} = \frac{(4m+4)}{(4m+1)^2}$$

$$\frac{u_m}{u_{m+1}} = \frac{6m^2 + 16 + 32m}{16m^2 + 1 + 8m}$$

$$\frac{u_m}{u_{m+1}} = \frac{n^2 \left(16 + \frac{16}{n^2} + \frac{32}{n} \right)}{n^2 \left(16 + \frac{1}{n^2} + \frac{8}{n} \right)} = \frac{16 + \frac{16}{\infty^2} + \frac{32}{\infty}}{16 + \frac{1}{\infty^2} + \frac{8}{\infty}}$$

$$= \frac{16}{16} \cdot \lim_{n \rightarrow \infty} \frac{u_m}{u_{m+1}} = 1 \quad \text{Test fail}$$

Apply Ratio test

$$\frac{u_m}{u_{m+1}} = \frac{(4m+4)}{(4m+1)^2}$$

$$n \left(\frac{(4n+4)^2}{(n+1)} - 1 \right)$$

$$n \left(\frac{16n^2 + 16 + 32n}{16n^2 + 1 + 8n} - 1 \right)$$

$$n \left(\frac{16n^2 + 16 + 32n - 16n^2 - 1 - 8n}{16n^2 + 1 + 8n} \right)$$

$$n \left(\frac{15 - 24n}{16n^2 + 1 + 8n} \right)$$

~~$$n \left(\frac{15 - 24n}{16n^2 + 1 + 8n} \right)$$

$$= \frac{15}{16} + \frac{1}{8n} + \frac{8}{n^2}$$

$$n \rightarrow \infty \Rightarrow \frac{15}{16} + 0 + 0 = 0.9375$$

$$16$$

$$0.9375 + 0.0625 = 1$$~~

$$n \left(\frac{15 + 24n}{16n^2 + 1 + 8n} \right)$$

$$n \left(\frac{15}{n} + 24 \right)$$

$$n \left(16 + \frac{1}{n^2} + \frac{8}{n} \right)$$

$$\frac{(15 + 24n)}{n}$$

$$16 + 0 + 0 = 16$$

$$= 1.5$$

$1 > 1$ so sum is 0.9375

Q Test the Convergence

* Logarithmic Test

If $\sum u_n$ is a positive term series such that

$\lim_{n \rightarrow \infty} n \cdot \log \frac{u_n}{u_{n+1}} = l$ the for $l > 1$, $\sum u_n$ is ~~gt~~ $l < 1$, $\sum u_n$ is ~~gt~~

① D'Alembert ratio Test fails

② $\lim_{n \rightarrow \infty} \left(\frac{u_n}{u_{n+1}} \right)$ involves e

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = e = 2.718$$

MMVVVI

pyq

Test the Convergence (MBT-2) 8 marks

$$2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \dots \infty$$

ans $u_n = \frac{2 \cdot n}{n!}$ $u_{n+1} = \frac{2 \cdot (n+1)}{(n+1)!}$

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \frac{2 \cdot n}{n!} \times \frac{(n+1)!}{2 \cdot (n+1)} \\ &= \frac{n \cdot (n+1)!}{n! \cdot (n+1) \cdot 2} = \frac{n \cdot (n+1) \cdot n!}{n! \cdot (n+1) \cdot 2} = \frac{n}{2} \end{aligned}$$



$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1 + \frac{1}{n}\right)^n \cdot x}$$

$$= \frac{n^n}{n^n \left(1 + \frac{1}{n}\right)^n \cdot x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n \cdot x} = \frac{1}{e x}$$

$$\left. \begin{array}{l} \frac{1}{e x} > 1, C \\ \frac{1}{e x} < 1, D \\ \frac{1}{e x} = 1 \end{array} \right\} \begin{array}{l} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \\ \text{Test fail} \end{array}$$

$$\boxed{x = \frac{1}{e}}$$

Apply logarithmic Test

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1 + \frac{1}{n}\right)^n}$$

taking log both sides

$$\log \left(\frac{u_n}{u_{n+1}} \right) = \log \left(\frac{e}{\left(1 + \frac{1}{n}\right)^n} \right)$$

$$\log \left(\frac{u_n}{u_{n+1}} \right) = \log e - \log \left(1 + \frac{1}{n} \right)^n$$

$$\log \left(\frac{u_n}{u_{n+1}} \right) = \log e - n \log \left(1 + \frac{1}{n} \right)$$

$$\log\left(\frac{un}{un+1}\right) = 1 - n \log\left(1 + \frac{1}{n}\right)$$

$$\therefore \log\left(1 + \frac{1}{n}\right) = n \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots \right]$$

$$\therefore \log(1+q) = \left(q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \dots - \infty\right)$$

$$\log\left(\frac{un}{un+1}\right) = 1 - n \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \dots \right]$$

$$= 1 - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \dots\right)$$

$$= \left(\frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \dots\right)$$

$$\therefore \lim_{n \rightarrow \infty} n \log\left(\frac{un}{un+1}\right)$$

$$= \lim_{n \rightarrow \infty} n \left(\frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \dots \right)$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \dots \right)$$

$$\therefore \frac{1}{2} < 1 \quad \text{QED}$$

(4M)

Cauchy Integral Test

(2x2)

Statement $\Rightarrow$ If $\forall x > 1$, $f(x)$ is
non-negative
($f(x) \geq 0$)

monotonically decreasing function
of x such that

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \text{ then } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges and } \int_1^{\infty} f(x) dx$$

Convergence or divergence together

$$\int_1^{\infty} f(x) \cdot dx = \text{finite} \quad \text{--- (Cgt)}$$

non finite --- (Dgt)

Q

Test the convergence

of $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)}$ using Cauchy Integral
test

$$\lim_{n \rightarrow \infty} \frac{1}{(x^2+1)} = f(x)$$

$$f(x) = \frac{1}{(x^2+1)} \quad f(x) > 0$$

pyo Test the convergence of the series

$$\sum_{n=1}^{\infty} n e^{-n^2} \text{ using Cauchy integral test}$$

Soln: $u_n = n e^{-n^2} = f(x)$

$$u_n = f(x)$$

$$f(x) = x \cdot e^{-x^2}$$

$$\int_1^{\infty} x \cdot e^{-x^2} \cdot dx = \left[\frac{x}{-2x} \cdot \frac{e^{-x^2}}{1} - \int \frac{-1}{4x^2} \cdot e^{-x^2} \right]_1^{\infty}$$

$$= \left[-\frac{1}{2} e^{-x^2} + \frac{1}{4x^2} e^{-x^2} \right]_1^{\infty}$$

$$\int_1^{\infty} x \cdot e^{-x^2} \cdot dx \text{ put } x^2 = t, 2x = \frac{dt}{dx}$$

$$\int_1^{\infty} x \cdot e^{-t} \cdot \frac{dt}{2x} \quad \left| \begin{array}{l} t^2 = t \\ t = \infty \end{array} \right| \quad dx = \frac{dt}{2x}$$

$$= \frac{1}{2} \int_1^{\infty} e^{-t} \cdot dt \Rightarrow \frac{1}{2} \left[\frac{e^{-t}}{-1} \right]_1^{\infty}$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-1} - \frac{e^{-1}}{-1} \right] \Rightarrow \frac{1}{2} \left[-\frac{1}{e} + \frac{1}{e} \right]$$

$$\Rightarrow \frac{1}{2} \left[-\frac{1}{\infty} + \frac{1}{e} \right] \Rightarrow \frac{1}{2} \left[-0 + \frac{1}{e} \right] = \frac{1}{2e}$$

finite

Q. Using Cauchy integral test, discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Sol: $u_n = \frac{n}{(n^2+1)^2} = f(x)$ $\because x^2+1=t$
 $2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

$f(x) = \frac{x}{(x^2+1)^2}$ $t=2$
 $\infty^2+1=t, t=\infty$

$\int_1^{\infty} \frac{x}{(x^2+1)^2} \cdot dx$ let $x^2+1=t$
 $2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$

$\int_1^{\infty} \frac{x}{t^2} \cdot \frac{dt}{2x}$

$\int_2^{\infty} \frac{dt}{2t^2} = \frac{1}{2} \int_2^{\infty} \frac{dt}{t^2} = \frac{1}{2} \left[-\frac{1}{t} \right]_2^{\infty}$

$\Rightarrow \frac{1}{2} \left[\left(-\frac{1}{\infty} \right) - \left(-\frac{1}{2} \right) \right]$

$\Rightarrow \frac{1}{2} \left[0 + \frac{1}{2} \right] = \frac{1}{4}$ finite

$\int_1^{\infty} f(x) \cdot dx$ converges and hence by integral test $\sum u_n$ converges.

Q. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ using Cauchy integral test.

pyq

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ using Cauchy integral test.

Solⁿ ÷

$$u_n = \frac{1}{(n+1)^2} = f(x) \quad \left| \begin{array}{l} x+1 = t \\ 1+1 = t \Rightarrow t=2 \\ \infty+1 = t \\ t = \infty \end{array} \right.$$

$$f(x) = \frac{1}{(x+1)^2}$$

$$\int_1^{\infty} \frac{1}{(x+1)^2} \cdot dx \quad \text{let } x+1 = t$$

$$dx = dt \Rightarrow dx = dt$$

$$\int_2^{\infty} \frac{1}{t^2} \cdot dt = \left[-\frac{1}{t} \right]_2^{\infty} = \left[\left(-\frac{1}{\infty} \right) - \left(-\frac{1}{2} \right) \right]$$

$$\Rightarrow \left[0 + \frac{1}{2} \right] = \frac{1}{2} \text{ finite}$$

$\int_1^{\infty} f(x) dx$ converges and hence by integral test $\sum u_n$ converges.

pyq

Test the convergence of series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ using Cauchy integral test.

Solⁿ ÷

$$u_n = \frac{1}{n^2+1}, = f(x) \Rightarrow f(x) = \frac{1}{x^2+1}$$

$$\int_1^{\infty} \frac{1}{x^2+1} \cdot dx, \left[\tan^{-1} x \right]_1^{\infty} \Rightarrow \left[\tan^{-1} \infty - \tan^{-1} 1 \right]$$

$$\Rightarrow \left[\frac{\pi}{2} - \frac{\pi}{4} \right] = \frac{\pi-1}{4} = \frac{\pi}{4} \text{ finite}$$

$\int_1^{\infty} f(x) \cdot dx$ converges and hence by integral test $\sum u_n$ converges.

pyq

$$Ex-2 \rightarrow 'C.I.T' \quad \sum_{n=2}^{\infty} \frac{1}{n \sqrt{n-1}}$$

$$u_n = \frac{1}{n \sqrt{n-1}} = f(x)$$

$$f(x) = \frac{1}{x \sqrt{x-1}}$$

$$= \int_2^{\infty} \frac{1}{x \sqrt{x-1}} \cdot dx \Rightarrow$$

$$\rightarrow \int_{\sqrt{3}}^{\infty} \frac{1}{x \sqrt{x-1}} \cdot \frac{\sqrt{x-1}}{x} \cdot dt$$

$$\int_{\sqrt{3}}^{\infty} \frac{1}{x^2} \cdot dt$$

$$\int_{\sqrt{3}}^{\infty} \frac{1}{1+t^2} \cdot dt$$

$$\left[\tan^{-1} t \right]_{\sqrt{3}}^{\infty}$$

$$\left[\tan^{-1} \infty - \tan^{-1} \sqrt{3} \right]$$

$$\left[\frac{\pi}{2} - \frac{\pi}{3} \right] = \frac{3\pi - 2\pi}{6} = \frac{\pi}{6} \text{ finite}$$

$\int_2^{\infty} f(x) \cdot dx$ Converges hence by integral test
 $\sum u_n$ Converges

$$\text{let } \sqrt{x^2-1} = t$$

$$x^2-1 = t^2$$

$$\sqrt{x^2-1} = t$$

$$\frac{1}{\sqrt{x^2-1}} \cdot x \cdot 2x = \frac{dt}{dx}$$

$$\boxed{dx = \frac{\sqrt{x^2-1} \cdot dt}{x}}$$

$$x^2-1 = t^2$$

$$\infty^2-1 = t^2$$

$$\infty = t^2 \quad t = \sqrt{x}$$

$$2^2-1 = t^2$$

$$t^2 = 3, \quad t = \sqrt{3}$$

$$\boxed{x^2 = 1+t^2}$$

Ex-7) Test the Convergence of the Series

$$\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2}$$

$$u_n = \frac{8 \tan^{-1} n}{1+n^2} = f(x) = \frac{8 \tan^{-1} x}{1+x^2}$$

$$\int_1^{\infty} 8 \tan^{-1} x \cdot \left(\frac{1}{1+x^2} \right) dx$$

$$\therefore \int f(x) \cdot f'(x) \cdot dx = \frac{f(x)^2}{2}$$

$$\Rightarrow \frac{8}{2} \left[\tan^{-1} x \right]_1^{\infty}$$

$$\Rightarrow 4 \left[\tan^{-1} x \right]_1^{\infty} \Rightarrow 4 \left[\tan^{-1} \infty - \tan^{-1} 1 \right]$$

$$\Rightarrow 4 \left[\left(\frac{\pi}{2} \right) - \left(\frac{\pi}{4} \right) \right] = 4 \left(\frac{\pi}{4} - \frac{\pi}{8} \right)$$

$$\Rightarrow 4 \left(\frac{4\pi - \pi}{8} \right) = 4 \left(\frac{3\pi}{8} \right) = \frac{3\pi}{2}$$

finite

Sum is Converges

Ques) ~~Test CGT of the series~~ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ C2.7

(p.xd)
$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{\sqrt{n}}$$

ans) $u_n = \frac{1}{\sqrt{n}}, u_{n+1} = \frac{1}{\sqrt{n+1}}$

$$u_n - u_{n+1} > 0$$

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}(\sqrt{n+1})}$$

$$= \frac{\sqrt{n} \left(\sqrt{1 + \frac{1}{n}} - 1 \right)}{\sqrt{n}(\sqrt{n+1})}$$

$$= \frac{\sqrt{1 + \frac{1}{n}} - 1}{\sqrt{n+1}} > 0$$

Let $\lim_{n \rightarrow \infty} u_n = \frac{1}{\sqrt{n}} = \frac{1}{\infty} = \frac{1}{\infty} = 0$

$$u_n = 0$$

So it is cg

(p.xd) Give an example of a series which is conditionally convergent but not absolutely convergent.

Sol: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots - \infty$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{p-1}}$$

(p.xd) for what value of p, does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or diverge?

Sol: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is said to be convergent if $p > 1$, and divergent if $p \leq 1$.

P48

Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$

$$u_n \Rightarrow u_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$v_n = \frac{1}{n^2} \quad \left[u_n = \frac{1}{n^2} \right]$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{(2n-1)}{n(n+1)(n+2)} \times n^2$$

$$= \frac{n^2(2n-1)}{n(n^2+n+2)} = \frac{n^2(2n-1)}{n^3+n^2+n+2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} \Rightarrow \frac{n^2 \cdot n \left(2 - \frac{1}{n} \right)}{n^3 \left(1 + \frac{2}{n} + \frac{1}{n} + \frac{2}{n^2} \right)}$$

$$= \frac{2 - \frac{1}{\infty}}{1 + \frac{2}{\infty} + \frac{1}{\infty} + \frac{2}{\infty^2}} = \frac{2-0}{1+0+0+0} = \frac{2}{1} = 2$$

finite
 $\lim$ will be considered

finite
 non zero

$$v_n = \frac{1}{n^2}$$

$$p=2$$

$$p \geq 1$$

(Cgt)

P49

Test the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^n$$

$$u_n = \left(\frac{n}{n+1} \right)^{n^2} \Rightarrow (u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1} \right)^n$$

$$\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n$$

$$(u_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n(1 + \frac{1}{n})} \right)^n$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-n}$$

$$\therefore \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$$

$$= \frac{1}{e} < 1$$

So CDT

Ques Test for convergence of $\sum \left(\frac{n}{n+1} \right)^{n^2}$

$$\text{Ans} \Rightarrow u_n = \left(\frac{n}{n+1} \right)^{n^2} \quad (u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1} \right)^n$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \Rightarrow \lim_{n \rightarrow \infty} \left(\frac{n}{n(1 + \frac{1}{n})} \right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-n} = \frac{1}{e} < 1$$

So by Cauchy root test
it is CDT

prob = Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(n+1) \cdot x^n}{n^3}$$

Sol: $u_n = \frac{n+1 \cdot x^n}{n^3}$ $u_{n+1} = \frac{(n+2) x^{n+1}}{(n+1)^3}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1) x^n}{n^3} \times \frac{(n+1)^3}{(n+2) x^{n+1}}$$

$$= \frac{(n+1)^2}{n^3 (n+2) \cdot x} \Rightarrow \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{n^3 \cdot n \left(1 + \frac{2}{n}\right) x}$$

$$\frac{u_n}{u_{n+1}} = \frac{1 + \frac{4}{n} + \frac{6}{n^2} + \frac{4}{n^3} + \frac{1}{n^4}}{\left(1 + \frac{2}{n}\right) \cdot x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1 + \frac{4}{\infty} + \frac{6}{\infty^2} + \frac{4}{\infty^3} + \frac{1}{\infty^4}}{\left(1 + \frac{2}{\infty}\right) x}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{1 \cdot x} = \frac{1}{x}$$

Now $\frac{1}{x} > 1 \Rightarrow$ Cgt

$\frac{1}{x} < 1 \Rightarrow$ Lgt

$\frac{1}{x} = 1 \Rightarrow$ test fail

*

Cauchy root test

if $\sum u_n$ is a +ve term series such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|u_n|} = l$$

then for $l < 1$, $\sum u_n$ is Cgt

$l > 1$, $\sum u_n$ is Dgt

$l = 1$, $\sum u_n$ Test fails

Q

Test the convergence :-

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$$

Sol: $u_n = \left(1 + \frac{1}{n}\right)^{-n^2} = (u_n)^{\frac{1}{n}} = \left(1 + \frac{1}{n}\right)^{-n}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$e = 2.7$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e}$$

$$= \frac{1}{e} < 1, \text{ Cgt}$$

$$\frac{2m}{4m}$$

* Leibnitz Test (on alternating series)

$\therefore$ An infinite of the form:

$$\sum_{n=1}^{\infty} (-1)^n \cdot u_n = (u_1 - u_2 + u_3 - u_4 + \dots - \infty)$$

(Alternating (+, -, +, -))
is said to be convergent if

① $u_n > u_{n+1}$ ② $\lim_{n \rightarrow \infty} u_n = 0$

Q Test the Convergence =

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$u_n = \frac{1}{n^2}$$

$$u_{n+1} = \frac{1}{(n+1)^2}$$

$$u_n - u_{n+1} > 0$$

$$\frac{1}{n^2} - \frac{1}{n^2+1+2n} = \frac{n^2+1+2n - n^2}{(n^2)(n^2+1+2n)}$$

$$\frac{1+2n}{(n^2+2n+1) \cdot n^2} = \frac{2n+1}{(n^2+2n+1)n^2} > 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

So it is correct

Alternative series में काल
यही test लगाना

Q Test the convergence:

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{(n^2+1)} \Rightarrow \cos n\pi = (-1)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2+1)}$$

$$u_n = \frac{(-1)^n}{(n^2+1)}$$

$$u_{n+1} = \frac{(-1)^{n+1}}{((n+1)^2+1)}$$

$$u_n - u_{n+1} = \frac{(-1)^n}{(n^2+1)} - \frac{(-1)^{n+1}}{((n+1)^2+1)}$$

$$= \frac{1}{(n^2+1)} + \frac{1}{((n+1)^2+1)}$$

$$= \frac{((n+1)^2+1) + (n^2+1)}{(n^2+1)((n+1)^2+1)}$$

$$= \frac{(n^2+2n+2) + (n^2+1)}{(n^2+1)((n+1)^2+1)}$$

$$= \frac{2n^2+2n+3}{(n^2+1)((n+1)^2+1)}$$

$$u_n = \frac{1}{n^2+1}$$

$$u_{n+1} = \frac{1}{(n+1)^2+1}$$

$$u_n - u_{n+1} = \frac{1}{(n^2+1)} - \frac{1}{((n+1)^2+1)}$$

$$= \frac{((n+1)^2+1) - (n^2+1)}{(n^2+1)((n+1)^2+1)}$$

$$= \frac{(n^2+2n+2) - (n^2+1)}{(n^2+1)((n+1)^2+1)}$$

$$= \frac{2n+1}{(n^2+1)((n+1)^2+1)}$$

$$\frac{n^2 + 1 + 2n + 1 - n^2 - 1}{(n^2 + 1)(n+1)^2 + 1}$$

$$= \frac{2n+1}{(n^2+1)(n+1)^2+1} > 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = \frac{1}{\infty^2+1} = 0$$

So it is cgt

* Absolute and Conditional

If a convergent series remains convergent after making all its terms +ve then it is said to be an absolutely cgt series otherwise it is said to be conditionally cgt series. $\sum_{n=1}^{\infty} u_n \rightarrow \sum$

① Test for Absolute Convergence Conditional Convergence.

Sol $\sum_{n=1}^{\infty} \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right)$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{It is convergent}$$

~~②~~ $\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \quad p=2 \quad p > 1 \quad \text{Convergent absolute convergent}$$

Q Test for absolute Conditional convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}$$

Solns $\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right)$

Q for what values of x does the following series converges?

$$x = -1$$

$$x = 1$$

$$\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right)$$

following

4 marks

M8T-2

$$\sum |u_n| = \sum \frac{|x|^n}{n}$$

$$\lim_{n \rightarrow \infty} \frac{|u_n|}{|u_{n+1}|}$$

$$u_n = \frac{|x|^n}{n}$$

$$u_{n+1} = \frac{|x|^{n+1}}{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{|x|^n}{n} \times \frac{n+1}{|x|^{n+1}}$$



$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n|x|} = \frac{n(1 + \frac{1}{n})}{n|x|}$$

$$\lim_{n \rightarrow \infty} \frac{|u_n|}{|u_{n+1}|} = \frac{(1 + \frac{1}{n})}{|x|} = \frac{1+0}{|x|} = \frac{1}{|x|}$$

if $\frac{1}{|x|} > 1$, Cgt

$\frac{1}{|x|} < 1$, dgt

$\frac{1}{|x|} = 1$, 1st fail

$|x| = 1$ when $x = 1$
the series becomes $1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1}{4}$

$(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty)$ which is

an alternative series and it is convergent

$$u_n - u_{n+1} > 0$$

$$\frac{|x|^n}{n} - \frac{|x|^{n+1}}{n+1} = \frac{n - (n+1)|x|}{(n+1)n} > 0$$

$$\lim_{n \rightarrow \infty} \frac{|x|^n}{n} = \frac{\infty}{\infty} = 0 = 0$$

$$= \frac{1}{0} \times \frac{0}{1} = 0$$

So it is convergent

when $x = -1$ the series becomes

$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \dots \infty$

$$-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \infty\right)$$

$$-\left(\sum \frac{1}{n^p}\right) \quad \sum = \frac{1}{n} \quad p=1$$

$$p < 1 \text{ diverges}$$

Q Test for absolute conditional convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^{n-1}}$$

Sol:

$$u_n = \frac{(-1)^{n-1}}{e^{n-1}} \rightarrow |u_n| = \frac{1}{e^{n-1}}$$

$$u_{n+1} = \frac{1}{e^{(n+1)-1}} = \frac{1}{e^{n+2-1}}$$

$$u_{n+1} = \frac{1}{e^{n+1}}$$

$$u_n - u_{n+1} \geq 0 \Rightarrow \frac{1}{e^{n-1}} - \frac{1}{e^{n+1}}$$

$$= \frac{e^{n+1} - e^{n-1}}{(e^{n-1})(e^{n+1})} = \frac{2}{(e^{n-1})(e^{n+1})} > 0$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n \left(2 - \frac{1}{n}\right)} = \frac{1}{\infty \left(2 - \frac{1}{\infty}\right)} = 0 \quad \text{convergent}$$

$$u_n = \frac{1}{e^{n-1}}, \quad v_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{e^{n-1}} \times n = \frac{n}{n \left(2 - \frac{1}{n}\right)} = \frac{1}{2 - \frac{1}{\infty}} = \frac{1}{2-1}$$

$$= \frac{1}{2} \cdot \text{finite}$$

$$\sqrt[n]{n} = \frac{1}{n} \quad p = 1$$

It is divergent

$\therefore$ it is conditionally convergence.

Q49 Test the convergence of the series using Cauchy integral test

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Soln: $u_n = \frac{1}{n(n+1)} = f(x) = \frac{1}{x(x+1)}$

$$\int_1^{\infty} \frac{1}{x(x+1)} dx = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{(x+1)} \right) dx$$

$$\Rightarrow \left[\log x - \log(x+1) \right]_1^{\infty}$$

$$\Rightarrow \left[(\log \infty - \log(\infty+1)) - (\log 1 - \log(1+1)) \right]$$

$$\Rightarrow \left[(\log \infty - \log \infty) - (0 - \log 2) \right]$$

$$\Rightarrow +\log 2 = 0.3010 = \text{finite}$$

So $\int_1^{\infty} f(x) dx$ converges and hence by

integral test sum converges.

Improper Integral

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Gamma function : Definition of Gamma

fmⁿ →

$$\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx$$

Note

$$\frac{-\infty}{e} = \frac{1}{\infty}$$

= 0

$$\frac{\infty}{e} = \infty$$

property -1 →

$$\Gamma(1) = 1$$

$\Gamma(1)$

$$\Gamma = \int_0^{\infty} e^{-x} x^{1-1} \cdot dx$$

$$\int_0^{\infty} e^{-x} \cdot dx \Rightarrow \left[\frac{e^{-x}}{-1} \right]_0^{\infty}$$

$$= \left(\frac{e^{-\infty}}{-1} \right) - \left(\frac{e^{-0}}{-1} \right)$$

$$= 0 - (-1) = 1$$

* Reduction formula for Gamma function

$$\Gamma(1) = 1$$

$$\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx$$

$$\int u \cdot v = u \int v \cdot dx - \int \left(\frac{du}{dx} \right) \left(\int v \cdot dx \right) \cdot dx$$

$$= \left[x \left(\frac{e^{-x}}{-1} \right) - \int (n-1) \cdot x^{n-2} \left(\frac{e^{-x}}{-1} \right) dx \right]_0^{\infty}$$

$$\Rightarrow 0 + (n-1) \int_0^{\infty} e^{-x} \cdot x^{n-2} \cdot dx$$

$$\Rightarrow (n-1) \cdot \Gamma(n-1) \Rightarrow \Gamma(n) = (n-1) \Gamma(n-1)$$

Property - 2

P.Y.Q

Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ Sol: $\therefore$

$$\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx$$

let

$$x = y$$

diff. w.r.t. to x

$$n x^{n-1} = \frac{dy}{dx} \Rightarrow dx \cdot n = dy$$

$$\boxed{\frac{n-1}{x} dx = \frac{dy}{n}}$$

$$\Gamma(n) = \int_0^{\infty} e^{-x} \cdot \frac{dy}{n}$$

$$\text{Now } \therefore x^n = y$$

apply exponential both sides

$$e^x = e^y = e^{\frac{y}{n}} = e^{\frac{y}{1/n}}$$

$$\Gamma(n) = \int_0^{\infty} e^{-y/n} \cdot \frac{dy}{n} \Rightarrow \Gamma(n) = \frac{1}{n} \int_0^{\infty} e^{-y/n} \cdot dy$$

$$n \Gamma(n) = \int_0^{\infty} e^{-y/n} \cdot dy$$

$$\therefore n = \frac{1}{2}$$

$$\frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-y^{1/2}} \cdot dy \Rightarrow \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-y^2} \cdot dy$$

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-y^2} \cdot dy \quad \text{--- (A)}$$

Similarly

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-x^2} \cdot dx \quad \text{--- (B)}$$

Multiply eqⁿ (A) x (B)

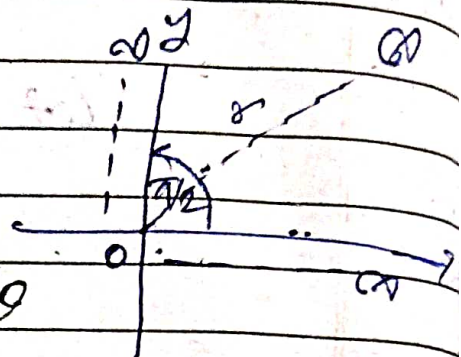
$$\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}} = 4 \int_0^{\infty} \int_0^{\infty} \frac{-(x^2+y^2)}{e} dx \cdot dy$$

change into polar

$$x^2 + y^2 = r^2$$

$$\therefore x = r \cos \theta, y = r \sin \theta$$

$$dx dy = r dr d\theta$$



$$\left(\sqrt{\frac{1}{2}}\right)^2 = 4 \int_0^{\pi/2} \int_0^{\infty} \frac{-r^2}{e} \cdot r dr d\theta$$

$$\text{put } r^2 = p \quad 2r dr = \frac{dp}{d\theta}$$

$$2r dr = dp$$

$$r dr = \frac{dp}{2}$$

$$= 4 \int_0^{\pi/2} \int_0^{\infty} \frac{-p}{e} \cdot \frac{dp}{2} \cdot d\theta$$

$$= \frac{4}{2} \int_0^{\pi/2} \int_0^{\infty} (-p \cdot dp) d\theta$$

$$= 2 \int_0^{\pi/2} \left[\frac{-p^2}{2} \right]_0^{\infty} d\theta$$

$$\left(\sqrt{\frac{1}{2}}\right)^2 = 2 \int_0^{\pi/2} 1 \cdot d\theta$$

$$= 2 \left[\theta \right]_0^{\pi/2}$$

$$= 2 \left[\frac{\pi}{2} - 0 \right]$$

$$= \pi \times 2$$

$$\left(\sqrt{\frac{1}{2}}\right)^2 = 2 \int_0^{\pi/2} \left(\left[\frac{-p^2}{2} \right]_0^{\infty} - \left[\frac{-p^2}{2} \right]_0^0 \right) d\theta$$

$$\left(\sqrt{\frac{1}{2}}\right)^2 = \pi$$

$$= 2 \int_0^{\pi/2} (0 + 1) d\theta$$

$$\boxed{\sqrt{\frac{1}{2}} = \sqrt{\pi}} \text{ proved}$$

$$\Gamma(n) = (n-1)! \quad \text{if } n = \text{integer}(+ve)$$

Beta - function

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$B(1, 1) = \int_0^1 x^{1-1} (1-x)^{1-1} dx$$

$$= \int_0^1 1 \cdot (1-x)^0 dx$$

$$= \int_0^1 1 \cdot dx = [x]_0^1$$

$$= 1 - 0$$

$$= 1$$

$$B(2, 1) = \int_0^1 x^{2-1} (1-x)^{1-1} dx$$

$$= \int_0^1 x \cdot dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

Reflection Property Proof

$$B(m, n) = B(n, m)$$

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{Put } (1-x) = t$$

$$x = 1-t$$

$$\frac{dx}{dt} = -1$$

$$dx = -dt$$

$$-dx = dt$$

$$\Rightarrow \int_0^1 (1-t) \cdot t^{m-1} dt = \int_1^0 t^{m-1} \cdot (1-t) (-dt)$$

$$= B(m, m)$$

* Relation between beta β & Γ (gamma)

$$B(m, m) = \frac{\Gamma(m) \cdot \Gamma(m)}{\Gamma(m+m)}$$

$$\beta(5, 3) = \frac{\Gamma(5) \cdot \Gamma(3)}{\Gamma(5+3)}$$

$$= \frac{\Gamma(5) \cdot \Gamma(3)}{\Gamma(8)}$$

$$= \frac{24 \times 2}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}$$

$$= \frac{1}{35 \times 3}$$

$$= \frac{1}{105}$$

(pyq) prove that $\beta(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

proof: we know that $\Gamma(n) = \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx$

by property of gamma function

$$\frac{\Gamma(n)}{z^n} = \int_0^{\infty} \frac{e^{-x} \cdot x^{n-1}}{z^n} \cdot dx$$

$$\Gamma(n) = z^n \int_0^{\infty} \frac{e^{-zx} \cdot x^{n-1}}{z^n} \cdot dx$$

$$\Gamma(n) = \int_0^{\infty} z^n \cdot \frac{e^{-zx} \cdot x^{n-1}}{z^n} \cdot dx$$

Multiply both sides by $(e^{-z} \cdot z^{m-1})$ we get

$$\Gamma(n) \cdot \frac{e^{-z} \cdot z^{m-1}}{z^n} = \int_0^{\infty} \frac{e^{-z} \cdot z^{m-1}}{z^n} \cdot \frac{e^{-zx} \cdot x^{n-1}}{z^n} \cdot dx$$

$$\Gamma(n) \cdot \frac{e^{-z} \cdot z^{m-1}}{z^n} = \int_0^{\infty} \frac{e^{-z(1+x)} \cdot z^{m+n-1} \cdot x^{n-1}}{z^n \cdot z^n} \cdot dx$$

Integrating both sides w.r. to 'z' b/w the limits 0 to ∞ , we get

$$\Gamma(n) \int_0^{\infty} \frac{e^{-z} \cdot z^{m-1}}{z^n} \cdot dz = \int_0^{\infty} \int_0^{\infty} \frac{e^{-z(1+x)} \cdot z^{m+n-1} \cdot x^{n-1}}{z^n \cdot z^n} \cdot dx \cdot dz$$

$$\Gamma(n) \cdot \Gamma(m) = \int_0^{\infty} \frac{z^{m-1}}{z^n} \cdot dx \int_0^{\infty} \frac{e^{-z(1+x)} \cdot z^{m+n-1}}{z^n \cdot z^n} \cdot dz$$

$$\Gamma(n) \cdot \Gamma(m) = \int_0^{\infty} \frac{z^{m-1}}{z^n} \cdot dx \cdot \frac{\Gamma(m+n)}{(1+x)^{m+n}} \quad \left\{ \because \frac{\Gamma(n)}{z^n} = \int_0^{\infty} \frac{e^{-zx} \cdot x^{n-1}}{z^n} \cdot dx \right\}$$

$$\frac{\Gamma(n) \cdot \Gamma(m)}{\Gamma(m+n)} = \int_0^{\infty} \frac{z^{m-1}}{(1+z)^{m+n}} \cdot dx \int_0^{\infty} \frac{z^{m-1}}{(1+z)^{m+n}} \cdot dx = \beta(m, m) \& \beta(m, m) = \beta(m, m)$$

$$\frac{\Gamma(n) \cdot \Gamma(m)}{\Gamma(m+n)} = \beta(m, m)$$

pyq prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
 Solⁿ we know that relationship b/w Beta & gamma funⁿ.

$$\beta(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} \quad \text{--- (1)}$$

by property of Beta function

$$\beta(m, n) = 2 \int_0^{\pi/2} (\sin \theta)^{m-1} \cdot (\cos \theta)^{n-1} \cdot d\theta \quad \text{--- (2)}$$

by comparing eqⁿ (1) & (2)

$$\frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} = 2 \int_0^{\pi/2} (\sin \theta)^{m-1} \cdot (\cos \theta)^{n-1} \cdot d\theta$$

put $m = \frac{1}{2}$, $n = \frac{1}{2}$ we get

$$\frac{\Gamma(\frac{1}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2})} = 2 \int_0^{\pi/2} (\sin \theta)^{(\frac{1}{2}-1)} \cdot (\cos \theta)^{(\frac{1}{2}-1)} \cdot d\theta$$

$$\frac{(\Gamma(\frac{1}{2}))^2}{\Gamma(1)} = 2 \int_0^{\pi/2} \sin^0 \theta \cdot \cos^0 \theta \cdot d\theta$$

$\because \Gamma(1) = 1$

$$(\Gamma(\frac{1}{2}))^2 = 2 \int_0^{\pi/2} 1 \cdot d\theta \Rightarrow (\Gamma(\frac{1}{2}))^2 = 2 \left[\theta \right]_0^{\pi/2}$$

$$\Rightarrow (\Gamma(\frac{1}{2}))^2 = 2 \left[\frac{\pi}{2} - 0 \right]$$

$$(\Gamma(\frac{1}{2}))^2 = \pi \times \cancel{x}$$

$$(\Gamma(\frac{1}{2}))^2 = \pi \Rightarrow \boxed{\Gamma(\frac{1}{2}) = \sqrt{\pi}}$$

Raushan
Kumar

pyq

prove that $\int_0^{\pi/2} \sin^p x \cdot \cos^q x \cdot dx = \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{q+1}{2})}{2 \cdot \Gamma(\frac{p+q+2}{2})}$ Hence find $\Gamma(\frac{1}{2})$

Sol: put $\sin^2 x = z \Rightarrow \sin x = \sqrt{z}$ $\sin x = z^{\frac{1}{2}}$
 diff w.r.to x

$2 \sin x \cdot \cos x = \frac{dz}{dx}$

$dx = \frac{dz}{2 \sin x \cdot \cos x}$

$\begin{cases} \because \sin^2 x = z \\ \cos^2 x = 1 - \sin^2 x = (1-z) \end{cases}$

$= \int_0^{\pi/2} (\sin^{\frac{p}{2}} x) \cdot (\cos^{\frac{q}{2}} x) \cdot dx \Rightarrow \int_0^{\pi/2} (z)^{\frac{p}{2}} \cdot (1-z)^{\frac{q}{2}} \cdot \frac{dz}{2 \sin x \cdot \cos x}$

$\because z = \sin^2 x$

when $x = 0$ then $z = 0$

when $x = \frac{\pi}{2}$ then $z = 1$

$\int_0^1 z^{\frac{p}{2}} \cdot (1-z)^{\frac{q}{2}} \cdot \frac{dz}{2 \cdot z^{\frac{1}{2}} \cdot (1-z)^{\frac{1}{2}}} = \int_0^1 \frac{z^{\frac{p}{2}} \cdot (1-z)^{\frac{q}{2}}}{2 \cdot z^{\frac{1}{2}} \cdot (1-z)^{\frac{1}{2}}} \cdot dz$

$= \frac{1}{2} \int_0^1 z^{\frac{p}{2} - \frac{1}{2}} \cdot (1-z)^{\frac{q}{2} - \frac{1}{2}} \cdot dz$

$= \frac{1}{2} \int_0^1 z^{\frac{p-1}{2}} \cdot (1-z)^{\frac{q-1}{2}} \cdot dz \Rightarrow \frac{1}{2} \int_0^1 z^{(\frac{p-1}{2} + 1 - 1)} \cdot (1-z)^{(\frac{q-1}{2} + 1 - 1)} \cdot dz$

$= \frac{1}{2} \int_0^1 z^{(\frac{p-1+2}{2} - 1)} \cdot (1-z)^{(\frac{q-1+2}{2} - 1)} \cdot dz \Rightarrow \frac{1}{2} \int_0^1 z^{(\frac{p+1}{2} - 1)} \cdot (1-z)^{(\frac{q+1}{2} - 1)} \cdot dz$

$\because \beta(m, n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} \cdot dx$

$= \frac{1}{2} \int_0^1 z^{(\frac{p+1}{2} - 1)} \cdot (1-z)^{(\frac{q+1}{2} - 1)} \cdot dz =$



$$\therefore \beta(m, m) = \frac{\sqrt{m} \cdot \sqrt{m}}{\sqrt{m+m}}$$

~~2~~

$$\frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad \therefore \beta(m, m) = \frac{\sqrt{m} \cdot \sqrt{m}}{\sqrt{m+m}}$$

$$\frac{\frac{1}{2} \times \sqrt{\frac{p+1}{2}} \cdot \sqrt{\frac{q+1}{2}}}{\sqrt{\frac{\frac{p+1}{2} + \frac{q+1}{2}}{2}}}$$

$$\frac{\frac{1}{2} \sqrt{\frac{p+1}{2}} \cdot \sqrt{\frac{q+1}{2}}}{\frac{1}{2} \sqrt{\frac{p+1}{2} + \frac{q+1}{2}}} = \frac{\sqrt{\frac{p+1}{2}} \cdot \sqrt{\frac{q+1}{2}}}{\sqrt{\frac{p+q+2}{2}}}$$

$$\int_0^{\pi/2} \sin^p x \cdot \cos^q x \cdot dx = \frac{\sqrt{\frac{p+1}{2}} \cdot \sqrt{\frac{q+1}{2}}}{2 \cdot \sqrt{\frac{p+q+2}{2}}} \quad \begin{array}{l} \text{proved} \\ \text{Now put} \\ p=q=0 \end{array}$$

$$\int_0^{\pi/2} 1 \cdot 1 \cdot dx = \frac{\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}}}{2 \cdot 1} \Rightarrow \left[x \right]_0^{\pi/2} = \frac{\left(\sqrt{\frac{1}{2}} \right)^2}{2}$$

$$\frac{\pi}{2} = \frac{\left(\sqrt{\frac{1}{2}} \right)^2}{2}$$

$$\frac{\pi}{2} \times 2 = \left(\sqrt{\frac{1}{2}} \right)^2 \Rightarrow \pi = \left(\sqrt{\frac{1}{2}} \right)^2$$

$$\boxed{\sqrt{\frac{1}{2}} = \sqrt{\pi}}$$

$$* \int_0^{\pi/2} \sin^{m-1} \theta \cdot \cos^{n-1} \theta \cdot d\theta$$

So, %

from L.H.S $\int_0^{\pi/2} \sin^{m-1} \theta \cdot \cos^{n-1} \theta \cdot d\theta$

let $x = \sin^2 \theta$ diff wr. to θ

$$\frac{dx}{d\theta} = 2 \sin \theta \cdot \cos \theta \quad [dx = 2 \sin \theta \cdot \cos \theta \cdot d\theta]$$

$$\int_0^{\pi/2} \sin^{m-1} \theta \cdot \cos^{n-1} \theta \cdot d\theta$$

$$\int_0^{\pi/2} (\sin^2 \theta)^{\frac{m-1}{2}} \cdot (1 - \sin^2 \theta)^{\frac{n-1}{2}} \cdot d\theta$$

$$\begin{aligned} x &= \sin^2 \theta \\ x &= \sin^2 \theta \\ x &= 0 \\ x &= \sin^2 \theta \\ x &= \frac{\pi}{2} \\ dx &= 2 \sin \theta \cdot \cos \theta \cdot d\theta \end{aligned}$$

$$\int_0^{\pi/2} \sin^{m-2} \theta \cdot \cos^{n-2} \theta \cdot d\theta$$

$$\int_0^{\pi/2} \sin^{m-1} \theta \cdot \cos^{n-1} \theta \cdot d\theta$$

Hence R.H.S proved

$$= \int_0^1 z^{\frac{p}{2}-1} (1-z)^{\frac{q}{2}-1} dz$$

$$= \frac{1}{2} \int_0^1 z^{\frac{p-1}{2}} (1-z)^{\frac{q-1}{2}} dz = \frac{1}{2} \int_0^1 z^{\frac{p-1}{2}+1-1} (1-z)^{\frac{q-1}{2}+1-1} dz$$

$$= \frac{1}{2} \int_0^1 x^{m-1} (1-x)^{n-1} dx = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

~~in computer~~

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)$$

$$\Rightarrow \int_0^{\pi/2} \sin^p x \cdot \cos^q x \, dx$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\Gamma\left(\frac{p+q+2}{2}\right)$$

$$\text{put } p=q=0$$

$\pi/2$

PRO Prove that $\int_0^{\pi/2} \tan \theta \, d\theta = \frac{\sqrt{3}}{4} \frac{\pi}{4}$

Sol $\int_0^{\pi/2} \frac{\sin^{1/2} \theta}{\cos^{1/2} \theta} \cdot d\theta = \int_0^{\pi/2} \sin^{1/2} \theta \cdot \cos^{-1/2} \theta \cdot d\theta$

by property of gamma funⁿ.

$$\int_0^{\pi/2} \sin^p x \cdot \cos^q x \cdot dx = \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{p+q+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1/2+1}{2}\right) \cdot \Gamma\left(\frac{-1/2+1}{2}\right)}{2 \Gamma\left(\frac{1/2-1/2+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right)}{2 \Gamma\left(\frac{\pi}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{1}{4}\right)}{2 \cdot 1}$$

$$\boxed{\int_0^{\pi/2} \tan \theta \cdot d\theta = \frac{\sqrt{3}}{4} \frac{\pi}{4}} \quad \text{proved}$$

$$\int_0^{\pi/2} 1 \cdot dx = \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{2 \Gamma\left(\frac{1}{2}\right)}$$

$$\left[x \right]_0^{\pi/2} = \frac{\left(\frac{1}{2}\right)^2}{2} \Rightarrow \frac{\pi}{2} = \frac{\left(\frac{1}{2}\right)^2}{2} = \frac{1}{2} \times \left(\frac{1}{2}\right)^2$$

$$\pi = \left(\frac{1}{2}\right)^2 = \frac{1}{2} = \sqrt{\pi}$$

Define Beta and gamma function

Beta function: The beta function denoted by $B(m, n)$ defined for positive real numbers

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx$$

Beta function is symmetric means

$$B(m, n) = B(n, m)$$

Gamma function: The gamma function denoted as Γn for positive real number n , it is defined as

$$\Gamma n = \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx$$

AI2

$$\int_0^{\infty} 1 \cdot dx = \frac{\Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{2 \Gamma(1)}$$

$$[x]_0^{\infty} = \frac{\Gamma\left(\frac{1}{2}\right)^2}{2} = \frac{\pi}{2}$$

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \pi = \Gamma\left(\frac{1}{2}\right)^2$$

find $\int_0^{\infty} \frac{-\sqrt{x}}{e} \cdot x^{1/4} \cdot dx$

$$\Rightarrow \int_0^{\infty} e^{-t} \cdot t^{n-1} dt = \Gamma(n)$$

put $\sqrt{x} = t$ $x = t^2$
 $\frac{dx}{dt} = 2t$ $dx = 2t dt$

$$= \int_0^{\infty} e^{-t} \cdot (t^2)^{1/4} \cdot 2t dt$$

$$\begin{aligned} \sqrt{x} &= t \\ \sqrt{0} &= t \\ 0 &= t \\ \sqrt{\infty} &= t \\ t &= \infty \end{aligned}$$

(3)

12

$$\int_0^{\infty} e^{-t} \cdot \left(\frac{t^2}{2}\right) \cdot 2t \, dt \quad \because \int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx$$

$$= 2 \int_0^{\infty} e^{-t} \cdot \left(\frac{t^3}{6}\right) \cdot dt$$

$$\Rightarrow 2 \int_0^{\infty} e^{-t} \cdot t^{(5/2-1)} \cdot dt$$

$$\Rightarrow 2 \Gamma\left(\frac{5}{2}\right)$$

$$\left[\frac{5}{2}-1 = \frac{3}{2}\right]$$

$$= \Gamma(n) = (n-1) \Gamma(n-1)$$

$$= 2 \Gamma\left(\frac{5}{2}\right) = 2 \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right)$$

$$= 2 \times \frac{3}{2} \left(\frac{1}{2} \Gamma\left(\frac{1}{2}\right) \right) = \frac{3}{2} \Gamma$$

$$\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{3}{2} \sqrt{\pi}$$

qm



Relation between beta and gamma funⁿ.

$$T.p \quad \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} = B(m, n)$$

(12 Nov 2020) आ सकदा है आगर
आपका सवाल कि चल रहा है

Sol $\Gamma(m) = \int_0^{\infty} e^{-t} \cdot t^{m-1} \cdot dt$

put $t = x^2$

$dt = 2x dx$

$$\Gamma(m) = \int_0^{\infty} e^{-x^2} \cdot (x^2)^{m-1} \cdot (2x dx)$$

$(-em - 2 + 1) = (em - 1)$

$$\Gamma(m) = 2 \int_0^{\infty} e^{-x^2} \cdot x^{2m-1} \cdot dx \quad \text{--- (1)}$$

similarly

$$\Gamma(m) = 2 \int_0^{\infty} e^{-y^2} \cdot y^{2m-1} \cdot dy \quad \text{--- (2)}$$

multiply (1) and (2)

$$\Gamma(m) \cdot \Gamma(m) = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} \cdot x^{2m-1} \cdot y^{2m-1} \cdot dx dy \quad \text{--- (3)}$$

let $x = r \cos \theta$

$y = r \sin \theta$

$x^2 + y^2 = r^2$ $dx dy = r dr d\theta$

$$\Gamma(m) \cdot \Gamma(m) = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} (r \cos \theta)^{2m-1} (r \sin \theta)^{2m-1} \cdot r dr d\theta$$

$$\Rightarrow f(m) \cdot f(m) = 4 \int_0^{\pi/2} \int_0^{\infty} \frac{r^{2m-1}}{e^r} (\cos \theta) (\sin \theta) \cdot dr d\theta$$

$$= 4 \left(\int_0^{\pi/2} \cos \theta \cdot \sin \theta d\theta \right) \left(\int_0^{\infty} \frac{r^{2m-1}}{e^r} dr \right)$$

$$\therefore f(m) = 2 \int_0^{\infty} \frac{r^{2m-1}}{e^r} \cdot dr \quad \text{--- (1)}$$

from eq (1)

$$\Rightarrow \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{p+q+2}{2}\right)}$$

$$= 2 \int_0^{\pi/2} \sin^p x \cdot \cos^q x dx$$

$$\beta: \quad p = 2m-1$$

$$q = 2n-1$$

$$\beta(m, n) \Gamma(m+n)$$

$$\Rightarrow 2 \int_0^{\pi/2} \sin^p x \cdot \cos^q x dx = \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \beta\left(\frac{2m-1+1}{2}, \frac{2n-1+1}{2}\right)$$

$$\beta(n, m) = \beta(m, n)$$

* Improper integrals

Q Evaluate $\lim_{N \rightarrow \infty} \int_0^N \frac{1}{1+x^2} \cdot dx$

$$= \lim_{N \rightarrow \infty} \left[\tan^{-1} x \right]_0^N \Rightarrow \left[\tan^{-1} \infty - \tan^{-1} 0 \right]$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} \quad \text{finite}$$

convergent

if finite
convergent
Non finite
divergent

Q2 $\int_{-a}^a \frac{dx}{\sqrt{a^2 - x^2}} = \left[\sin^{-1} \frac{x}{a} \right]_{-a}^a$

$$= \left[\sin^{-1} \frac{a}{a} - \sin^{-1} \frac{-a}{a} \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \text{finite}$$

convergent

Q3 Indeterminate forms :

Q4 $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$

$$\lim_{x \rightarrow 0} \frac{0-0}{0} = \frac{0}{0}$$

L'HOSPITAL

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$$

$$= \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

$$\lim_{x \rightarrow 0} = \frac{1 - \cos x}{3x^2}$$

$$\lim_{x \rightarrow 0} = \frac{1 - 1}{0} = \frac{0}{0}$$

again apply L-Hospital rule

$$\lim_{x \rightarrow 0} \frac{0 + \sin x}{6x}$$

$$\lim_{x \rightarrow 0} = \frac{0 + \cos x}{6}$$

$$= \frac{1}{6} \text{ finite}$$

so it is convergent

Q 88

$$\lim_{x \rightarrow \pi/2} (\sec x - \tan x)$$

$$\lim_{x \rightarrow \pi/2} (\sec x - \tan x) = \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$\lim_{x \rightarrow \pi/2} \left(\frac{1 - \sin x}{\cos x} \right)$$

$$\lim_{x \rightarrow \pi/2} \left(\frac{0 - \cos x}{-\sin x} \right)$$

$$\cos x =$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{0 + \cos x}{\sin x}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\sin x} = 0$$

finite $\rightarrow$ convergent

अब वल
zero
आए व
अ व अ

~~WQ~~ $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}$ let $(1+x)^{\frac{1}{x}} = y$

$$\lim_{x \rightarrow 0} \log(1+x)^{\frac{1}{x}} = \log y$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \log(1+x) = \log y$$

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \log y$$

$$\frac{g(x)}{f(x)} \quad \frac{g'(x)}{f'(x)}$$

~~$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} =$$~~

$$\lim_{x \rightarrow 0} \frac{1}{(1+x)} = \log y$$

~~$$1 = \log y$$~~

$$e = e^{\log y}$$

$$e = y \quad y = e \text{ finite convergent}$$

Note $\Rightarrow$ जब भी किसी के पावर

$\frac{1}{x}$ हो तो $\log$ लिखा जाता है

(2) जब भी $\frac{0}{0}$ या ∞ हो तो L Hospital rule लागू

Q Evaluate :-

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} - \cot x \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right)$$

~~$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x \sin x} \right)$$~~

$$\lim_{x \rightarrow 0} \left(\frac{\sin x - x \cos x}{x \sin x} \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{0}{0} \right)$$

again L Hospital

$$\lim_{x \rightarrow 0} \left(\frac{\cos x - (x(-\sin x) + \cos x)}{x \cos x + \sin x} \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{\cos x + x \sin x - \cos x}{x \cos x + \sin x} \right)$$

$$\lim_{x \rightarrow 0} \frac{x \sin x}{x \cos x + \sin x}$$

$$\lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x \sin x + \cos x + \cos x}$$

$$\lim_{x \rightarrow 0} \left(\frac{0 \times 1 + 0}{-0 \times 0 + 1 + 1} \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{0}{2} \right)$$

$$\lim_{x \rightarrow 0} 0 \quad \text{Convergent}$$

(pg) Evaluate $\lim_{x \rightarrow \infty} (1+x)^{1/x}$

Solⁿ $\lim_{x \rightarrow \infty} (1+x)^{1/x} = (1+\infty)^{\frac{1}{\infty}} = \infty^0 = \infty^0 = \infty$

~~Hospital Rule~~

$\lim_{x \rightarrow \infty} (1+x)^{1/x}$

Let $(1+x)^{1/x} = y$ Taking log b.s

$\log (1+x)^{1/x} = \log y$

$\frac{\log (1+x)}{x} = \log y$

$\frac{f(x)}{g(x)}$ apply L-Hos rule

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{1+x}}{1} = \log y = \frac{1}{1+\infty} = \log y = \frac{1}{\infty} = \log y$$

$$= \frac{0}{1} = \log y \Rightarrow 0 = \log y$$

$y = e^0$ finite

$y = 1$

Convergent

pg 8 State Taylor and Maclaurin Theorems

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AM Taylor Theorem

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n(x).$$

Maclaurin Theorem (a=0)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n(x), \quad R_n(x) = \frac{x^n}{n!} f^{(n)}(\xi)$$

pg 9 Expand $f(x) = \log x$ about $x=1$ using Taylor Theorem.

Sol: $f(x) = \log x$ | $f(1) = \log 1 = 0$ | $x=1 \Rightarrow x=1, a=1$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}$$

$$f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3}$$

$$f'''(1) = 2$$

$$f(x) = f(1) + \frac{(x-1)^1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \dots$$

$$= 0 + \frac{(x-1)(1)}{1} + \frac{(x-1)^2(-1)}{2} + \frac{(x-1)^3(2)}{6} + \dots$$

$$\log x = \left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots \right)$$

* Expand $f(x) = (2x^3 + 7x^2 + x - 1)$ in powers of $x-2$,

here $x-2=0 \quad x=2=a=2$

solⁿ $\rightarrow$

$$f(x) = 2x^3 + 7x^2 + x - 1 \quad f(2) = 16 + 28 + 2 - 1 = 45$$

$$f'(x) = 6x^2 + 14x + 1 \quad f'(2) = 24 + 28 + 1 = 53$$

$$f''(x) = 12x + 14 \quad f''(2) = 38$$

$$f'''(x) = 12 \quad f'''(2) = 12$$

$$f(x) = f(2) + \frac{(x-2)}{1!} f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2)$$

$$= 45 + \frac{(x-2)}{1} (53) + \frac{(x-2)^2}{2} (38) + \frac{(x-2)^3}{6} (12)$$

$$f(x) = \left[45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3 \right]$$

Q40) Expand $f(x) = \sin x$ using Maclaurin's theorem

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -1$$

$$a=0 \quad x=0=a=0$$

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \dots$$

$$f(x) = x - \frac{x^3}{3 \times 2 \times 1}$$

$$f(x) = x - \frac{x^3}{6}$$

$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\int_0^{\pi/2} \cot x \cdot dx = \log |\sin x|$$

$$= \left[\log |\sin \frac{\pi}{2}| - \log |\sin 0| \right]$$

$$= [\log |1| - \log |0|]$$

$$= [0 - \infty] = -\infty$$

infinite

So it is convergent

Improper अव आगे $\rightarrow$

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(py) Test the convergence of the improper integral $\int_0^{\infty} e^{-x} \cdot dx$

ans $\Rightarrow \int_0^{\infty} e^{-x} \cdot dx = \left[\frac{e^{-x}}{-1} \right]_0^{\infty}$

$$= \left[\left(\frac{e^{-\infty}}{-1} \right) - \left(\frac{e^{-0}}{-1} \right) \right] = \left[\infty - 1 \right]$$

$= \infty$ infinite
divergence

~~(py) Evaluate $\int_0^{\pi/2} \tan \theta \cdot d\theta$~~

~~Sol $\Rightarrow \int_0^{\pi/2} \tan \theta \cdot d\theta = \int_0^{\pi/2} \sec \theta - \tan \theta \cdot d\theta = 1$~~

~~$\tan^2 \theta = \sec^2 \theta - 1$~~

~~$\tan \theta = \sec \theta - 1$~~

~~$\frac{1}{2} \times \frac{1}{2} \times 4$~~

(py) Evaluate the improper integral $\int_0^{\infty} \frac{e^{-2x}}{\sqrt{x}} \cdot dx$

Sol $\Rightarrow \int_0^{\infty} \frac{e^{-2x}}{\sqrt{x}} \cdot dx$

$$= \left[x^{\frac{5}{2}} \left(\frac{e^{-2x}}{-2} \right) - 5x^{\frac{3}{2}} \left(\frac{e^{-2x}}{4} \right) + 20x^{\frac{1}{2}} \left(\frac{e^{-2x}}{-8} \right) - 60 \left(\frac{e^{-2x}}{16} \right) + \right.$$

$$\left. 120x \left(\frac{e^{-2x}}{-32} \right) - 120 \left(\frac{e^{-2x}}{64} \right) \right]_0^{\infty} = \infty - \frac{1}{2} - \frac{5}{4} - \frac{20}{8} - \frac{60}{16}$$

infinite $\rightarrow$ divergence

~~$\frac{120}{32} + \frac{120}{32} - \frac{120}{64} = \infty$~~

(Q48) Test the Convergence or divergence of the improper integral $\int_0^{\infty} e^{-x} dx$ solved before

(Q48) Test the Convergence or divergence of the improper integral $\int_0^{\infty} \frac{dx}{9+x^2}$

$$\begin{aligned} \text{Sol}^n &: \int_0^{\infty} \frac{dx}{9+x^2} = \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^{\infty} \\ &= \left[\left(\frac{1}{3} \tan^{-1} \frac{\infty}{3} \right) - \left(\frac{1}{3} \tan^{-1} 0 \right) \right] \\ &= \left[\frac{1}{3} \times \frac{\pi}{2} - 0 \right] = \frac{\pi}{6} - 0 = \frac{\pi}{6} \\ &= \text{finite} \rightarrow \text{Convergence} \end{aligned}$$

(Q48) Evaluate improper integral $\int_0^{\infty} \frac{dx}{1+x^2}$ why is it convergent?

$$\begin{aligned} \text{Sol}^n &: \int_0^{\infty} \frac{dx}{1+x^2} = \left[\tan^{-1} x \right]_0^{\infty} \\ &= \left[\tan^{-1} \infty - \tan^{-1} 0 \right] = \frac{\pi}{2} - 0 = \frac{\pi}{2} \text{ finite} \end{aligned}$$

Convergence

It is convergent just because solⁿ is finite.

(Pys) Test for Convergence or divergence of the improper integral $\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}$

Sol: $\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2} = \left[\frac{1}{a} \tan^{-1} \frac{x}{a} \right]_{-\infty}^{\infty}$

$$= \left[\frac{1}{a} \tan^{-1} \frac{\infty}{a} - \frac{1}{a} \tan^{-1} \left(\frac{-\infty}{a} \right) \right]$$

$$\Rightarrow \frac{1}{a} \left(\frac{\pi}{2} \right) + \frac{1}{a} \left(\frac{\pi}{2} \right) = \frac{1}{a} \left[\pi \right] = \frac{\pi}{a} \text{ finite}$$

Convergent

~~Indeterminate form again~~

(Pys) Evaluate $\lim_{x \rightarrow 0} \frac{x \cos x - \log(1+x)}{x^2}$

Sol: $\lim_{x \rightarrow 0} \frac{0 \cos 0 - \log(1+0)}{0} = \frac{0}{0}$

Apply L-Hospital Rule

$$\lim_{x \rightarrow 0} \frac{-x \sin x + \cos x - \frac{1}{1+x}}{2x}$$

$$\lim_{x \rightarrow 0} = \frac{-0 \sin 0 + \cos 0 - \frac{1}{1+0}}{2(0)} = \frac{0}{0}$$

again apply L-Hospital rule $\frac{0-0+1}{2} = \frac{1}{2}$

$$\lim_{x \rightarrow 0} \frac{-x \cos x - \sin x + \frac{1}{(1+x)^2}}{2} = \text{finite Convergent}$$

Q10 Evaluate: $\lim_{x \rightarrow 0} (\sin x)^{\tan x}$

Sol: $\lim_{x \rightarrow 0} (\sin x)^{\tan x} = (\sin 0)^0 = 0^0$

let $y = (\sin x)^{\tan x}$

Taking log both s.

$$\log y = \log (\sin x)^{\tan x}$$

$$\log y = \tan x \cdot \log (\sin x)$$

$$\lim_{x \rightarrow 0} \log y = \lim_{x \rightarrow 0} \frac{\log (\sin x)}{\cot x}$$

$$\log y = \frac{\log (\sin 0)}{\cot 0} = \frac{\log (0)}{\infty} = \frac{0}{\infty}$$

~~apply~~ apply L-Hospital rule

$$\log y = \frac{\frac{1}{\sin x} \times \cos x}{-\cot^2 x} = \frac{\cot x}{-\cot^2 x}$$

$$\lim_{x \rightarrow 0} \log y = \frac{\cot x}{-\cot^2 x} = -\frac{1}{\cot x}$$

$$\lim_{x \rightarrow 0} \log y = -\sin x \cdot \cot x$$

$$\lim_{x \rightarrow 0} \log y = -\sin(0) \cdot \cot(0)$$

$$= -0 \times 1$$

$$\log y = 0$$

$$y = e^0 \quad [y=1] \text{ finite convergent}$$

(pyq) $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a}$

Solⁿ $\Rightarrow \lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \frac{0}{0}$

~~Apply L-Hospital Rule~~

~~$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$~~

$\lim_{x \rightarrow a} \frac{(x - a)(x^2 + ax + a^2)}{(x - a)}$

$\lim_{x \rightarrow a} x^2 + ax + a^2$

$\lim_{x \rightarrow a} a^2 + a^2 + a \times a = a^2 + a^2 + a^2$

$\lim_{x \rightarrow a} 3a^2$ finite
convergent

(pyq) Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x)$

Solⁿ $\lim_{x \rightarrow \frac{\pi}{2}} (\sec \frac{\pi}{2} - \tan \frac{\pi}{2}) = \infty - \infty = \infty$

~~Apply L-Hospital Rule~~

~~$\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$~~

$\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \frac{1 - \sin x}{\cos x}$

Apply L-Hospital Rule

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{0 - \cos x}{-\sin x}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} = \frac{\cos x}{\sin x} = \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} = \frac{0}{1} = 0$$

finite convergent

(Ques) Evaluate $\lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}}$

$$\text{Sol: } \lim_{x \rightarrow \infty} (e^x + \infty)^{\frac{1}{\infty}} = \infty^{\frac{1}{\infty}}$$

$$\text{Let } y = (e^x + x)^{\frac{1}{x}}$$

Taking log both sides

$$\log y = \frac{1}{x} \log(e^x + x)$$

$$\lim_{x \rightarrow \infty} = \frac{\log(e^x + \infty)}{\infty} = \frac{\infty}{\infty}$$

Apply L-Hospital rule

$$\lim_{x \rightarrow \infty} \frac{1}{e^x + x} \cdot e^x + 1 = \frac{e^x}{e^x + x}$$

$$\lim_{x \rightarrow \infty} = \frac{e^x}{e^x + \infty} = \frac{\infty}{\infty}$$

again apply L-Hospital rule

$$\lim_{x \rightarrow \infty} \frac{e^x}{x+1}$$

again apply L-Hospital rule

$$\lim_{x \rightarrow \infty} = \frac{\cancel{e^x}}{\cancel{x}} = \text{finite}$$

Convergent

* Taylor & Maclaurin

Ques Expand $\log(1+x)$ using Maclaurin's theorem.

$$\text{Sol}^n: f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \dots$$

$$f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \dots$$

$$f(x) = \log(1+x) \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \quad f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \quad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \quad f'''(0) = 2$$

$$\log(1+x) = 0 + \frac{x(1)}{1!} + \frac{x^2(-1)}{2!} + \frac{x^3(2)}{3!} + \dots$$

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots}$$

Ques Expand $\sin x$ in ascending powers of $(x - \frac{\pi}{2})$ using Taylor Theorem.

$$\text{Sol}^n: x = \frac{\pi}{2} \quad a = \frac{\pi}{2}$$

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

$$f(x) = \sin x \quad \left| \quad f(a) = 1 \right.$$

$$f(x) = \cos x \quad \left| \quad f'(a) = 0 \right.$$

$$f'(x) = -\sin x \quad \left| \quad f''(a) = -1 \right.$$

$$f''(x) = -\cos x \quad \left| \quad f'''(a) = 0 \right.$$

$$\sin x = 1 + \frac{(x-\frac{\pi}{2})}{1!} (0) + \frac{(x-\frac{\pi}{2})^2}{2!} (-1) + \frac{(x-\frac{\pi}{2})^3}{3!} (0) + \dots$$

$$\sin x = 1 + 0 + \frac{(x-\frac{\pi}{2})^2}{2!} + 0 + \dots$$

$$\boxed{\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2} + \dots}$$

Ques Find The Maclaurin's Series for $f(x) = \cos x$.

$$\text{Sol: } f(x) = f(0) + \frac{x f'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \dots$$

$a=0$ because Maclaurin's theorem will be applied

$$\begin{array}{l|l} f(x) = \cos x & f(0) = 1 \\ f'(x) = -\sin x & f'(0) = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \end{array}$$

$$\cos x = 1 + \frac{x^0}{1!} + \frac{x^2(-1)}{2!} + \frac{x^3(0)}{3!} + \dots$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + \dots}$$

Q20 Expand $\tan x$ in powers of $(x - \frac{\pi}{4})$ upto first four terms.

$$\text{Sol}^m: f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$$x - \frac{\pi}{4} = a = \frac{\pi}{4}$$

$$f(x) = \tan x \quad f(a) = 1$$

$$f'(x) = \sec^2 x \quad f'(a) = (\sqrt{2})^2 = 2$$

$$f''(x) = 2 \sec x \cdot \sec x \cdot \tan x \quad f''(a) = 2 \times 2 \times 1 = 4$$

$$f'''(x) = 2[\tan x \cdot 2 \sec x \cdot \sec x + \sec^2 x] \quad f'''(a) = 2[2 \times 2 + 2]$$

$$f'''(a) = 2[4 + 2] = 16$$

$$\tan x = 1 + \frac{(x - \frac{\pi}{4})}{1!}(2) + \frac{(x - \frac{\pi}{4})^2}{2!}(4) + \frac{(x - \frac{\pi}{4})^3}{3!}(16) + \dots$$

$$\boxed{\tan x = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3 + \dots}$$

(PYQ) Obtain the first four terms of Taylor's series of $\cos x$ about $x = \frac{\pi}{4}$

Sol: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$

$$f(x) = \cos x \quad \left| \quad f(a) = \frac{1}{\sqrt{2}} \quad x = \frac{\pi}{4} = a = \frac{\pi}{4} \right.$$

$$f'(x) = -\sin x \quad \left| \quad f'(a) = -\frac{1}{\sqrt{2}} \right.$$

$$f''(x) = -\cos x \quad \left| \quad f''(a) = -\frac{1}{\sqrt{2}} \right.$$

$$f'''(x) = \sin x \quad \left| \quad f'''(a) = \frac{1}{\sqrt{2}} \right.$$

$$\cos x = \frac{1}{\sqrt{2}} + \frac{(x - \frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}})}{1!} + \frac{(x - \frac{1}{\sqrt{2}})^2 \times (-\frac{1}{\sqrt{2}})}{2!} + \frac{(x - \frac{1}{\sqrt{2}})^3 \times \frac{1}{\sqrt{2}}}{3!} + \dots$$

$$\cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{(x - \frac{1}{\sqrt{2}})}{1!} - \frac{1}{\sqrt{2}} \frac{(x - \frac{1}{\sqrt{2}})^2}{2!} + \frac{1}{\sqrt{2}} \frac{(x - \frac{1}{\sqrt{2}})^3}{6} + \dots$$

$$\cos x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{(x - \frac{1}{\sqrt{2}})}{1} - \frac{1}{\sqrt{2}} \frac{(x - \frac{1}{\sqrt{2}})^2}{2} + \frac{1}{\sqrt{2}} \frac{(x - \frac{1}{\sqrt{2}})^3}{6} + \dots$$

(pyq)

$$\sqrt{\frac{1}{2}} = \sqrt{\pi}$$

sol: We know that by gamma property

$$\frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} \cdot \cos \theta \cdot d\theta$$

① **p-Series Test** $\div$ An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is said to be
cgt if $\boxed{p > 1}$ and dgt if $\boxed{p \leq 1}$

② **Comparison Test** $\div$ An infinite series in which all the term of $\sum u_n$ is convergent
positive
if $\frac{u_n}{v_n} \leq k$ & divergent if $\frac{u_n}{v_n} \geq k$

③ **D'Alembert ratio test** $\div$ If $\sum u_n$ is a +ve term series such that

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{u_{n+1}} \right) = l \quad \begin{array}{l} \text{then } l > 1, \sum u_n \text{ is cgt} \\ l < 1, \sum u_n \text{ is dgt} \\ l = 1, \text{ Test fail} \end{array}$$

④ **Raabe's test** $\div$ If $\sum u_n$ is a positive (+ve) term series such that

$$\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = l \quad \begin{array}{l} \text{then } l > 1, \sum u_n \text{ is cgt} \\ l < 1, \sum u_n \text{ is dgt} \\ l = 1, \text{ Test fail} \end{array}$$

⑤ **Logarithmic Test** $\div$ If $\sum u_n$ is a +ve term series such that

$$\lim_{n \rightarrow \infty} n \cdot \log \left(\frac{u_n}{u_{n+1}} \right) = l \quad \begin{array}{l} \text{then } l > 1, \sum u_n \text{ cgt} \\ l < 1, \sum u_n \text{ dgt} \end{array}$$

(6) Cauchy root test : If $\sum u_n$ is a term series such that

$$\lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}} = l \text{ then } l < 1, \text{ cgt}$$

$$l > 1, \text{ dgt}$$

$l = 1$, Test fail.

(7) Leibnitz test : An infinite series of the form of $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is said to be

cgt if (1) $u_n > u_{n+1}$ (2) $\lim_{n \rightarrow \infty} u_n = 0$

(8) Cauchy Integral test : If for $x \geq 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(x) = u_n$, for all positive integral values of n , then the series $\sum u_n$ and the integral $\int_1^{\infty} f(x) dx$ converge or diverge

if $\int_1^{\infty} f(x) dx = \text{finite} \rightarrow \text{converge}$

if $\int_1^{\infty} f(x) dx = \text{infinite} \rightarrow \text{diverge}$

① State ~~the~~ Taylo

① State Taylor theorem: If $f(x)$ is a function of x which can be expanded in power of $(x-a)$. it is defined as

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots$$

② Maclaurin's Theorem: If $f(x)$ is a function of x , which can be expanded ~~in power~~ as a power series in x .

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$